# Algebra Qualifying Exam

## Fall 2000

Everyone must do two problems in each of the four sections. To pass at the PhD level, you must attempt at least three 20-point problems. On multiple part problems, do as many parts as you can; however, not all parts count equally.

### Groups

A1. (10 points) Let $D_{2n}$ be the dihedral group of order $2n$ with $n > 1$. Determine the number of subgroups of $D_{2n}$ of index 2, and justify your answer.

A2. (15 points) A group of order a power of a prime $p$ is called a $p$–group. Let $G$ be a finite group. Prove that for any given prime $p$, there exists a unique normal subgroup $N$ of $G$ such that (i) $G/N$ is a $p$–group and (ii) any homomorphism $\pi$ of $G$ into a $p$–group is trivial on $N$ (that is, $\pi(N) = 1$).

A3. (20 points) Let $G$ be a finite group of order $n$. Suppose that $G$ has a unique subgroup of order $d$ for each positive divisor $d$ of $n$. Prove that $G$ is a cyclic group.

## Rings

B1. (10 points) Let $M$ be a module over a commutative ring $A$. If every increasing (resp. decreasing) sequence of $A$–submodules of $M$ terminates after finite steps, the $A$–module $M$ is called noetherian (resp. artinian).

(a) Prove that the $\mathbf{Z}$–module $\mathbf{Z}$ is noetherian and non-artinian.

(b) Prove that the $\mathbf{Z}$–module $\bigcup_{n=1}^{\infty}(p^{-n}\mathbf{Z}/\mathbf{Z})$ is artinian and non-noetherian.

B2. (15 points) Let $A$ be a commutative ring with identity. Suppose that $a \in A$ is not nilpotent (that is, $a^n \neq 0$ for all $n > 0$).

(a) Prove that there exists a prime ideal $P \subset A$ such that $a \notin P$;

(b) Give an example of a ring $A$ and a non-nilpotent $a \in A$ such that $a$ is contained in $M$ for all maximal ideals $M \subset A$. Justify your example.

B3. (20 points) Let $A$ be a commutative noetherian ring with identity $1 \neq 0$. Write $X(\mathfrak{a})$ for the set of prime ideals of $A$ containing a given ideal $\mathfrak{a}$. Suppose that $X(0) = X(\mathfrak{a}) \cup X(\mathfrak{b})$ and $X(\mathfrak{a}) \cap X(\mathfrak{b}) = \emptyset$ for two ideals $\mathfrak{a}$ and $\mathfrak{b}$. Prove the following facts:

(a) $A = \mathfrak{a} + \mathfrak{b}$;

(b) $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a}\mathfrak{b}$;

(c) The ideal $\mathfrak{a}\mathfrak{b}$ consists of nilpotent elements.
Hint: You may use the assertion of B2 (a);

(d) There exists a positive integer $n$ such that $A$ is isomorphic to the product ring $(A/\mathfrak{a}^n) \times (A/\mathfrak{b}^n)$.

## Fields

C1. (10 points) Find the minimal polynomial of $\sqrt{2} + \sqrt[3]{3}$ over the field of rational numbers $\mathbf{Q}$.

C2. (15 points) Let $a$ be a primitive 16-th root root of unity over the field $F$. Determine the dimension $[F(a) : F]$ when $F$ is:

a) The field of 9 elements,

b) The field of 7 elements,

c) The field of 17 elements,

d) What can you say in the case $p = 2$?

C3. (20 points) Let $F$ be an infinite field of characteristic $p > 0$. Recall that a finite dimensional extension $L/F$ is said to be *simple* if $L = F(u)$ for some element $u \in L$.

a) Suppose $L/F$ has a finite number of intermediate fields. Show that $L/F$ is simple.

b) Let $K$ be an intermediate field $F \subset K \subset L$, and suppose that $L = F(u)$ with $x^r + a_1 x^{r-1} + \cdots + a_r$ the monic irreducible polynomial of $u$ over $K$, $a_i \in K$. Show that $K = F(a_1, a_2, \ldots, a_r)$.

c) Conclude that $L/F$ is simple if and only if there are a finite number of intermediate fields.

d) Let $E = F(x, y)$ where $x$ and $y$ are indeterminates, and set

$$M = E(x^{1/p}, y^{1/p})$$

Show that $M/E$ has an infinite number of intermediate fields.

## Linear Algebra

D1. (10 points) Let $V$ be a finite dimensional vector space over a field $F$ and $T : V \to V$ a linear map whose characteristic polynomial has distinct roots, all in $F$. Show that $T$ is diagonalizable.

D2. (15 points) Describe all non-diagonalizable $4 \times 4$ matrices over $\mathbf{Q}$ such that $A^5 + A^2 = 0$ up to similarity. Justify your answer.

D3. (20 points) Let $V$ be a complex vector space with positive definite inner product $( \, , \, )$, and $T : V \to V$ a linear map. Recall that the adjoint $T^*$ of $T$ is defined by:

$$(T(x), y) = (x, T^*(y))$$

for all $x, y \in V$, and that $T$ is called *normal* if $T$ and $T^*$ commute. Suppose that $T^3 = TT^*$. Let $U$ be the kernel of $T$ and $W$ the orthogonal complement of $U$ in $V$.

(a) Show that $W$ is $T^*$-invariant.

(b) Show that $U$ is $T^*$-invariant.

(c) Conclude that $W$ is $T$ invariant.

(d) Show that the restrictions of $T$ and $T^*$ to $W$ commute.

(e) Show that $T$ is normal.