Spring 2000

Algebra Qualifying Exam

Everyone must do two problems in each of the four sections.

To pass at the Ph.D. level, you must attempt at least three 20 point problems. On multiple part problems, do as many parts as you can; however, not all parts count equally.

A. GROUPS

A1.(10 points) State a theorem which classifies (i.e. lists) all finite abelian groups up to isomorphism. This means that each finite abelian group should be isomorphic to exactly one group of your list. Use your classification to list abelian groups of order 24.

A2. (15 points) Let S_5 be the symmetric group on 5 letters. For each positive integer n, list the number of elements of S_5 of order n. Justify your answer.

A3. (20 points) Let $\mathbb{F}_4$ be the field with 4 elements. Let $G = SL(2, \mathbb{F}_4)$ be the group of 2 by 2 invertible matrices with entries in $\mathbb{F}_4$. What is the order of G ? Show, by analysing the action of G on the lines containing the origin in $(\mathbb{F}_4)^2$, that G is a simple group.(Hint: how many lines containing the origin are there?)

B. RINGS

B1.(10 points) List, up to isomorphism, all commutative rings with 4 elements. Prove your answer.

B2.(15 points) Let p be a prime number. Show that a free $\mathbb{Z}$ module of rank 2 has p+1 submodules of index p.

B3.(20 points) Let R be a commutative noetherian ring in which each ideal I is principal and satisfies $I^2 = I$. Show that R is isomorphic to a finite product of fields.

C. FIELDS

C1. Let $\alpha = 1 + \sqrt[3]{2} + \sqrt[3]{4}$.

(a) Find the degree of α over $\mathbb{Q}$. Justify your answer.

(b) Find a normal closure of $\mathbb{Q}(\alpha)/\mathbb{Q}$. Justify your answer.

C2. Let q be a power of a prime integer, $n \in \mathbb{N}$. Let k be the least positive integer such that $q^k \equiv 1 \ (mod\ n)$. Prove that the finite field $\mathbb{F}_{q^k}$ is a splitting field of the polynomial $X^n - 1$ over $\mathbb{F}_q$.

C3. Find a subfield F in the field of rational functions $\mathbb{C}(X)$ such that $\mathbb{C}(X)/F$ is a Galois extension with the Galois group isomorphic to the symmetric group S_3. (Hint: Consider automorphisms of $C(X)$ given by $X \mapsto \frac{aX+b}{cX+d}, a, b, c, d \in \mathbb{Z}$.)

D. LINEAR ALGEBRA

D1. Let A be a linear operator on a vector space of dimension n such that A^m is the zero operator for some m.
(a) Prove that all eigenvalues of A are equal to zero.
(b) Prove that $A^n = 0$.

D2. Describe, up to similarity, all 4×4 matrices A over $\mathbb{Q}$ such that $A^5 = -A^3$ but $A^3 \neq 0$. Justify your answer.

D3. Let End(V) be the ring of all linear operators on a finite dimensional vector space V (with respect to the addition and composition of operators). For an operator $A \in$ End(V) let L_A be the subspace in End(V) generated by the powers A^i, $i \geq 0$.
(a) Show that L_A is a subring in End(V).
(b) Prove that if L_A is a field then the characteristic polynomial of A is a power of an irreducible polynomial.