Algebra Qualifying Exam

Fall 2001

Everyone must do two problems in each of the four sections.

To pass at the Ph.D. level, you must attempt at least three 20-point problems. On multiple part problems, do as many parts as you can; however, not all parts count equally.

Groups

G1. (10 points) Let G be a finite group whose center has index n. Show that every conjugacy class in G has at most n elements.

G2. (15 points) Let G be a subgroup of S_n that acts transitively on the set $\{1, 2, \ldots, n\}$. Let H be the stabilizer in G of an element $x \in \{1, 2, \ldots, n\}$. Prove that

$$\bigcap_{g \in G} gHg^{-1} = \{e\}$$

G3. (20 points) Let G be the group of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

where $a \in (\mathbf{Z}/p)^*$ and $a \in \mathbf{Z}/p$. Describe all normal subgroups of G. Hint: find a convenient normal subgroup of order p.

Rings

R1. (10 points) Let R be a commutative ring, $I \subset R$ a nonzero ideal. Prove that if I is a free R-module then $I = aR$ for an element $a \in R$ which is not a zero divisor in R. Hint: consider the rank of I.

R2. (15 points) (a) Give an example of prime ideal in a commutative ring that is not maximal.

(b) Let R be a commutative ring with identity. Suppose for every element $x \in R$ there exists an integer $n = n(x) > 1$ such that $x^n = x$. Show that every prime ideal in R is maximal.

R3. (20 points) Let R be a ring.

(a) Prove that if a is a nilpotent element in a ring R with identity, then the element $1 + a$ is invertible.

In the next two parts, let $f(X) = a_0 + a_1X + \cdots + a_nX^n$ be a polynomial in $R[X]$ of degree n, that is, $a_n \neq 0$.

(b) Show that if R is an integral domain, then $f(X)$ is invertible in $R[X]$ if and only if $n = 0$.

(c) Show that if R is a commutative ring, $f(X)$ is invertible in $R[X]$ if and only if all a_0 is invertible and a_i are nilpotent in R for every $i \geq 1$.

Fields

F1. (10 points) Let $f(x) = x^3 - 2x - 2$.

(a) Show that $f(x)$ is irreducible over $\mathbb{Q}$.

(b) Let θ be a complex root of $f(x)$. Express θ^{-1} as a polynomial in θ with coefficients in $\mathbb{Q}$.

F2. (15 points) Let $f(x) = x^3 + nx + 2$ where n is an integer. Determine the (infinitely many) values of n for which f is irreducible over $\mathbb{Q}$.

F3. (20 points) Let G be the Galois group of $x^p - 2$ over $\mathbb{Q}$ where p is a prime. Show that G is isomorphic to the group of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

where $a \in (\mathbb{Z}/p)^*$ and $a \in \mathbb{Z}/p$.

Linear Algebra

LA1. (10 points) Let T be a linear operator on a finite-dimensional vector space V such that $\mathrm{Im}(T)$ and $\mathrm{Im}(T^2)$ have the same dimension. Show that $\ker T \cap \mathrm{Im}(T) = 0$.

LA2. (15 points) Find all similarity classes of 4×4 matrices A over $\mathbb{Q}$ such that $A^2 \neq \pm A$ and $A^2 \neq I$ but $A^3 = A$ (I is the identity 4×4 matrix).

LA3. (20 points) Let V be a vector space over a field k. A bilinear form $f : V \times V \to k$ is called skew-symmetric if $f(u, v) = -f(v, u)$ for all $v, u \in V$ and is called alternating if $f(v, v) = 0$ for all $v \in V$.

(a) Prove that every alternating form is skew-symmetric.

(b) Give an example of a skew-symmetric form which is not alternating. Hint: choose k of characteristic 2.

(c) Show that all alternating forms on V form a vector space $Alt(V)$ and find $\dim Alt(V)$ if $\dim V = n$.