Spring 2001

Algebra Qualifying Exam

Everyone must do two problems in each of the four sections.

To pass at the Ph.D. level, you must attempt at least three 20-point problems. On multiple part problems, do as many parts as you can; however, not all parts count equally.

A. Groups.

A1. (10 points) Determine a complete set of groups of order eight up to isomorphism and show that every group of order eight is isomorphic to one of these.

A2. (15 points) A finite group G acts on itself by conjugation. Determine all possible G if this action yields precisely three orbits. Prove your result.

A3. (20 points) Let G be a finitely generated group.

a. Show for each integer n there exist finitely many subgroups of index n.

b. Suppose that there exists a subgroup of finite index in G. Prove that G contains a characteristic subgroup of finite index.

B. Rings.

B1. (10 points) A commutative ring R with unit is said to be a local ring if it has a unique maximal ideal. Show that a commutative ring R with unit is a local ring if and only if for any two elements $u, v \in R$ satisfying $u + v = 1$ at least one of u, v is a unit of R.

B2. (15 points) Let $R = \mathbf{R}[x, y]$. Find a finitely generated R-module M that is not a direct sum of cyclic R-modules, and prove that it is not.

B3. (20 points) Let $f_1(z_1, \ldots, z_n), f_2(z_1, \ldots, z_n), \ldots, f_n(z_1, \ldots, z_n)$ be n polynomials in $\mathbf{C}[z_1, \ldots, z_n]$. Assume that $f_i(0, 0, \ldots, 0) = 0$ for all $i = 1, \ldots, n$. Prove that the origin is the only point of $\mathbf{C}^n$ where all of the f_i vanish if and only if the ideal I generated by $f_1, \ldots, f_n$ contains all monomials of degree N for some sufficiently large N.

C. Fields.

C1. (10 points) Let F be a prime field, i.e., the rationals or a field with p elements. Prove that an algebraic closure of F has infinite degree over F.

[Hint: You may want to do the two cases separately.]

C2. (15 points) Let $f \in \mathbf{Q}[x]$ be a polynomial of degree three. Let $K = \mathbf{Q}(\theta)$ be a splitting field of f. Determine all the possible galois groups of K/F, prove these are all such, and give explicit examples of K, i.e., determine a θ or f.

C3. (20 points) Prove that the polynomial $x^4 + 1$ is not irreducible over any field of positive characteristic.

D. Linear Algebra.

D1. (10 points) Let $V = M_{n\times n}(\mathbf{R})$ be the $n \times n$ real matrices. If $A, B \in V$, define $< A, B >= \text{tr } (A^t B)$.

(a) Prove that $<,>$ is a positive-definite symmetric inner product on V.

(b) If $E_{i,j}$ is the matrix with a 1 in the i, j place and zeros elsewhere, prove that $\{E_{i,j} \mid 1 \le i, j \le n\}$ is an orthonormal basis for V.

D2. (15 points) Let k be a field. Prove that $\{x^i \otimes y^j \mid i, j \ge 0\}$ is a basis for the k-vector space $k[x] \otimes_k k[y]$. Use this to show that $k[x] \otimes_k k[y] \cong k[x, y]$ as vector spaces over k.

D3. (20 points) Let $SO(3) = \{A \in M_{3\times 3}(\mathbf{R}) \mid A^t A = I \text{ and } \det(A) = 1\}$.

(a) Prove if $A \in SO(3)$ then $+1$ is an eigenvalue of A.

(b) Prove that if W is the subspace of $\mathbf{R}^3$ orthogonal to a non-zero eigenvector of $A \in SO(3)$ with eigenvalue $+1$ then A takes W to W and acts as rotation by some angle θ on W.