Algebra Qualifying Exam – Fall 2002

Test Instructions: All problems are worth 20 points. You are expected to do two problems in each of the four sections. Your total score will be computed by dropping the lowest scoring problem in each section. In problems where arguments must be given, you will lose points if you fail to state clearly the basic results you use.

GROUP THEORY

PROBLEM 1.

a) Let A be a free abelian group of rank n. If H is a subgroup of A, show that H is free abelian of rank n if and only if A/H is finite.

PROBLEM 2.

Let G be a finite group of order 108. Show that G has a normal subgroup of order 9 or 27.

PROBLEM 3.

Let G be a finite group and P a p-Sylow subgroup. Let $N_G(P)$ be the normalizer of P in G. Show that:

a) P is the unique p-Sylow subgroup of $N_G(P)$ (Do not quote a theorem that this is true!)
b) $N_G(P)$ is self normalizing in G

RING THEORY

PROBLEM 1.

Let R be a commutative ring with 1, and let $S = R[x]$ be the polynomial ring in one variable. Suppose M is a maximal ideal of S. Prove that M cannot consist entirely of 0-divisors.

Hint: You may want to distiguish the cases $x \in M$ or $x \notin M$

PROBLEM 2.

Let R be a commutative ring with 1, and suppose I and J are ideals of R so that: $I + J = R$. Show that:

(i) $IJ = I \cap J$

(i) $R/IJ \cong R/I \oplus R/J$

PROBLEM 3.

Let R be a commutative ring with 1, and let $S = R[x]$ be the polynomial ring in one variable. Let $f \in S$. If f is a unit of S (that is, f is invertible in S), show that f has the form $f = u + g$ where u is a unit in R and $g \in S$ is a nilpotent element without constant term.

FIELDS

PROBLEM 1.

a) Determine the minimal polynomial of $u = \sqrt{3 + 2\sqrt{2}}$ over Q.
b) Determine the minimal polynomial of u^{-1} over Q.

PROBLEM 2.

a) Let F be the field generated by the roots of the polynomial $X^6 + 3$ over Q. Determine the Galois group of F/Q.
b) Describe all subfields of F.

PROBLEM 3.

Let p be a prime integer such that $p \equiv 2$ or $3 \pmod 5$. Prove that the polynomial

$$1 + X + X^2 + X^3 + X^4$$

is irreducible over Z/pZ.

LINEAR ALGEBRA

PROBLEM 1.

Let T be a linear operator on a finite-dimensional vector space V such that $\operatorname{im} T = \operatorname{im} T^2$. Prove that $\ker T = \ker T^2$. (im and ker refer to the image and kernel of T)

PROBLEM 2.

Determine, up to similarity, all 3×3 matrices A over Q such that $A^2 + 2A^3 + A^4 = 0$ but $A + A^2 \neq 0$.

PROBLEM 3.

Let $T_1, T_2, \ldots, T_m$ be linear operators on a vector space of dimension n. Assume that
(i) $\dim \operatorname{im}(T_i) = 1$ and
(ii) $T_i^2 \neq 0$ and $T_i T_j = 0$ for every $i \neq j$.
Prove that $m \leq n$.