Algebra Qualifying Exam (Spring 2002)

Test Instructions: All problems are worth 20 points. Your total score will be computed by dropping the the lowest scoring problem. In problems where arguments must be given, you will lose points if you fail to state clearly the basic results you use.

GROUP THEORY

PROBLEM 1.

Show that a group G of order $2m$, where m odd, has a normal subgroup of order m.

PROBLEM 2.

List, up to isomorphism, all finite abelian groups A satisfying the following two conditions:

(i) A is a quotient of $\mathbf{Z}^2$, and

(ii) A is annihilated by 18, i.e. $18a = e$ for all a in A.

Your list should contain a representative of each isomorphism class exactly once. How many groups are there?

PROBLEM 3.

Prove that a group G of order 120 is not simple.

RING THEORY

PROBLEM 1.

Let R be a ring and A and B be two *non-isomorphic* simple, left R-modules (a left-module is simple if it has no proper submodules, i.e., submodules other than $\{0\}$ and itself). Show that the only proper submodules of $M = A \oplus B$ are $\{(\alpha, 0) : \alpha \in A\}$ and $\{(0, \beta) : \beta \in B\}$.

PROBLEM 2.

Let R be a commutative local ring, that is, R has a unique maximal ideal M.

(i) Show that if x lies in M, then $1 - x$ is invertible.

(ii) Show that if R is Noetherian and I is an ideal satisfying $I^2 = I$, then $I = 0$. Hint: consider a minimal set of generators for I.

PROBLEM 3.

Let $\mathbf{F}_2$ be the field with 2 elements and let $R = \mathbf{F}_2[X]$. List, up to isomorphism, all R-modules of order 8.

LINEAR ALGEBRA

PROBLEM 1.

Let $\varphi : M_3(\mathbf{Q}) \to M_3(\mathbf{Q})$ be the map sending m to $\varphi(m) = m^2 + 3m + 3$. Show that $\varphi(m) \neq 0$ for all $m \in M_3(\mathbf{Q})$.

PROBLEM 2.

Let A be a real matrix with column vectors $A_1, A_2, ..., A_n$. If the A_j are mutually orthogonal, then

$$|\det A| = \prod_{j=1}^{n} |A_j|$$

This follows because $|\det({}^tA \cdot A)| = |\det A|^2$ and ${}^tA \cdot A$ is a diagonal matrix with diagonal entries $|A_1|^2, |A_2|^2, ..., |A_n|^2$. Prove that a general matrix satisfies the inequality

$$|\det A| \leq \prod_{j=1}^{n} |A_j|$$

Hint: apply the Gram-Schmidt orthogonalization process to the columns.

PROBLEM 3.

Let $T \in M_3(\mathbf{C})$ and let $\mathcal{A}_T$ be the centralizer of T in $M_3(\mathbf{C})$. Show that $\dim(\mathcal{A}_T) \geq 3$ and describe (up to similarity) the linear transformations T such that $\dim(\mathcal{A}_T) = 3$.

GALOIS THEORY

PROBLEM 1.

Let $\mathbf{F}_7$ be the field with 7 elements and let L be the splitting field of the polynomial $X^{171} - 1 = 0$ over $\mathbf{F}_7$. Determine the degree of L over $\mathbf{F}_7$, explaining carefully the principles underlying your computation.

PROBLEM 2.

Show that there exists a Galois extension of $\mathbf{Q}$ of degree p for each prime p. State precisely all results which are needed to jusfify your answer.

PROBLEM 3.

Let $\alpha = \sqrt{i+2}$ where $i = \sqrt{-1}$.

(a) Compute the minimal polynomial of α over $\mathbf{Q}$.

(b) Let F be the splitting field and compute the degree of F over $\mathbf{Q}$;

(c) Show that F contains 3 quadratic extensions of $\mathbf{Q}$;

(d) Use this information to determine the Galois group.