Algebra Qualifying Exam

Winter 2002

Everyone must do two problems in each of the four sections. If three problems of a section are tried, only two problems of highest score count (the lowest score is ignored). On multiple part problems, do as many parts as you can; however, not all parts count equally.

Groups

A1. Let G be a free abelian group of rank n for a positive integer n (therefore $G \cong \mathbb{Z}^n$ as groups).

(a) Prove for a given integer $m > 1$, there are only finitely many subgroups H of index m in G;

(b) Find a formula of the number of subgroups of G of index 3. Justify your answer.

A2. Prove or disprove: there exsits a finite abelian group G whose automorphism group has order 3.

A3. Let S and G be p–groups (with $G \neq \{e\}$), and assume that S acts on G by automorphisms. Show that the fixed subgroup $G^S = \{g \in G | s(g) = g \text{ for all } s \in S\}$ is non-trivial (i.e., is not the trivial subgroup $\{e\}$).

Rings

B1. Let F be a field and A be a commutative F–algebra. Suppose A is of finite dimension as a vector space of F.

(a) Prove all prime ideals of A are maximal. Hint: consider maps $R/P \to R/P$ (P prime) of the form $x \to ax$ with a in R.

(b) Prove that there are only finitely many maximal ideals of A.

B2. Let $A = M_n(F)$ be the ring of $n \times n$ matrices with entries in an infinite field F for $n > 1$. Prove the following facts:

(a) There are only 2 two-sided ideals of A;

(b) There are infinitely many maximal left ideals of A. Hint: show that $Ax = Ay$ ($x, y \in A$) if and only if $\mathrm{Ker}(x) = \mathrm{Ker}(y)$.

B3. Let $\mathbb{F}_2$ be the field with 2 elements and $A = \mathbb{F}_2[T, \frac{1}{T}]$ for an indeterminate T. Prove the following facts:

(a) The group of units in A is generated by T.

(b) There are infinitely many distinct ring endomorphisms of A.

(c) The ring automorphism group $\mathrm{Aut}(A)$ is of order 2.

Fields

C1. The discriminant of the special cubic polynomial $f(x) = x^3+ax+b$ is given by $-4a^3-27b^2$. Determine the Galois group of the splitting field of $x^3 - x + 1$ over

(a) $\mathbb{F}_3$, the field with 3 elements.

(b) $\mathbb{F}_5$, the field with 5 elements.

(c) $\mathbb{Q}$, the rational numbers.

C2. A field extension $K/\mathbb{Q}$ is called *biquadratic* if it has degree 4 and if $K = \mathbb{Q}(\sqrt{a}, \sqrt{b})$ for some $a, b \in \mathbb{Q}$.

(a) Show that a biquadratic extension is normal with Galois group $\mathrm{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and list all sub-extensions.

(b) Prove that if $K/\mathbb{Q}$ is a normal extension of degree 4 with $\mathrm{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ then $K/\mathbb{Q}$ is biquadratic.

C3. Let K be a finite extension of the field F with no proper intermediate subfields.

(a) If K/F is normal, show that the degree $[K;F]$ is a prime.

(b) Give an example to show that $[K;F]$ need not be prime if K/F is not normal, and justify your answer.

Linear Algebra

D1. Let $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ where I is the $n \times n$ identity matrix. Suppose that S is a $2n \times 2n$ *symplectic* matrix, meaning that S is real and satisfies ${}^tSJS = J$, where ${}^tS$ is the transpose of S.

(a) Show that ${}^tS$ is symplectic.

(b) Show that S is similar to S^{-1}.

(c) It is always true that $\det S = 1$. Prove this in case $n = 1$.

D2. Suppose that A is a linear operator on the vector space $\mathbb{C}^n$ and that $v \in \mathbb{C}^n$ satisfies $(A - aI)^2 v = 0$ for some $a \in \mathbb{C}$, so that v is a *generalized* eigenvector of A with eigenvalue a. Suppose that $|a| < 1$. Show that

$$\|A^m v\| \to 0$$

as $m \to \infty$, where $\|.\|$ is the Euclidean norm on $\mathbb{C}^n$.

D3. Let the $n \times n$ matrix A be defined over the field F. Suppose that A has finite order:

$$A^m = I$$

for some positive integer m.

(a) If the characteristic of F is 0, show that A may be diagonalized over F.

(b) Show that the conclusion of (a) is not true for an arbitrary field F.