Test Instructions: All problems are worth 20 points. You are expected to do two problems in each of the four sections. Your total score will be computed by dropping the lowest scoring problem in each section. In problems where arguments must be given, you will lose points if you fail to state clearly the basic results you use.

GROUP THEORY

PROBLEM 1.

Let G be a finitely generated group, and $n > 1$ an integer. Show that G has at most a finite number of subgroups of index n.

PROBLEM 2.

Let G be a finite group, K a normal subgroup, and P a p-Sylow subgroup of K for some prime p. Prove that $G = KN_G(P)$.

PROBLEM 3.

Suppose G is the free abelian group on generators x, y, z, w, considered as an additive group. Let $a = x - z + 2w$, $b = x - y + w$, $c = 3x - y - 2z + 5w$, $d = 2x - 2y + 4w$. If $H = \langle a, b, c, d \rangle$, determine the structure of G/H.

RING THEORY

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PROBLEM 1.

a)   Let R be a commutative ring with 1. Suppose $f \in R[x]$ is a non-zero 0-divisor in the polynomial ring $R[x]$. Assume that R has no non-zero nilpotent elements. Show there is a non-zero element $a \in R$ so that $a \cdot f = 0$.

b)   Give an example of an R and f so that all coefficients of f are 0-divisors in R, but f is not a 0-divisor in $R[x]$.

PROBLEM 2.

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Let R be a ring, not necessarily commutative, and M a Noetherian left R-module. Suppose $f : M \to M$ is a surjective R-module map from M to M. Prove that f is an isomorphism.

PROBLEM 3.

a)   Let R be a commutative ring with 1, and S a multiplicatively closed subset of R not containing 0. Suppose I is an ideal of R maximal with respect to exclusion of S (i.e. $I \cap S$ is empty and I is largest such). Prove that I is a prime ideal of R.

b)   Show that every prime ideal of R arises as in part a).

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FIELDS

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PROBLEM 1.

Determine the Galois group of the polynomial $X^4 + 3X^2 + 1$ over Q.

PROBLEM 2.

● Let $f(X)$ be a polynomial of degree $n > 0$ over a field F.

a)   Prove that there is a field homomorphism $\alpha : F(X) \to F(X)$ such that $\alpha(X) = f(X)$.

b)   Let L be the image of α. Prove that the field extension $F(X)/L$ is finite and find its degree.

c)   Find the minimal polynomial of X over L.

PROBLEM 3.

Let p be a prime integer. Suppose that the degree of every finite extension of a field F is divisible by p. Prove that the degree of every finite extension of F is a power of p.

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LINEAR ALGEBRA

PROBLEM 1.

Let A be a linear operator in a finite dimensional vector space. Prove that if $A^2 = A$ then $\mathrm{Trace}(A) = \mathrm{Rank}(A)$.

PROBLEM 2.

Let L/F be a field extension and let A and B be $n \times n$ matrices over F. Prove that if A and B are conjugate in $\mathrm{M}_n(L)$, then A and B are conjugate in $\mathrm{M}_n(F)$.

PROBLEM 3.

Let V be a finite dimensional vector space over a field F. Let B be a non-degenerate bilinear form on V. (For every nonzero $v \in V$ there is $u \in V$ such that $B(u, v) \neq 0$.)

a) For every $v \in V$ define a linear form $l_v : V \to F$ by $l_v(u) = B(u, v)$. Prove that the map $V \to V^*$ given by $v \mapsto l_v$ is an isomorphism of vector spaces.

b) Prove that for every linear operator a of V there is a linear operator a^* such that $B(a(u), v) = B(u, a^*(v))$ for all $u, v \in V$.

c) Prove that $(ab)^* = b^* a^*$ for every two linear operators a and b.

d) Suppose that B is either symmetric or skew-symmetric (that is $B(u, v) = B(v, u)$ or $B(u, v) = -B(v, u)$ respectively). Prove that $a^{**} = a$ for a linear operator a.