Algebra Qualifying Exam

Winter 2003

Everyone must do two problems in each of the four sections. If three problems of a section are tried, only two problems of highest score count (the lowest score is ignored). On multiple part problems, do as many parts as you can; however, not all parts count equally.

Groups

A1. List, up to isomorphism, all abelian groups A which satisfy the following three conditions:
  (i) A has 108 elements;
  (ii) A has an element of order 9;
  (iii) A has no element of order 24.

A2. Let $N \geq 1$ be a positive integer. Show that a finitely generated group G has only finitely many subgroups of index at most N.

A3. Let $N \geq 2$ be an integer. Show that a subgroup of index 2 in S_N is A_N. Here S_N and A_N are the symmetric and alternating groups for N, respectively.

Rings

B1. Give an example of two integral domains A and B which contain a field F such that $A \otimes_F B$ is not an integral domain. Justify your answer. Hint: Take A to be the field of rational functions $\mathbb{F}_p(X)$ for the field $\mathbb{F}_p$ with p elements.

B2. Let $\mathbb{F}_q$ be the finite field of q elements, and put $F = \mathbb{F}_q$ and $K = \mathbb{F}_{q^2}$. Write $\sigma : K \to K$ for the field automorphism given by $x^\sigma = x^q$. Let

$$B = \left\{ \left(\begin{smallmatrix} a & b \\ db^\sigma & a^\sigma \end{smallmatrix} \right) \middle| a, b \in K \right\}$$

for a given $d \in F^\times$. Prove the following three facts:

(a) B is a subalgebra of dimension 4 over F inside the F–algebra of 2×2 matrices over K.

(b) B is a division algebra if and only if there exists no $c \in K$ such that $d = cc^\sigma$.

(c) B cannot be a division algebra.

B3. Let A be a discrete valuation ring with maximal ideal M, and define

$$B = \left\{ (a, b) \in A \times A \middle| a \equiv b \mod M \right\}.$$

Prove the following facts:

(a) B has only one maximal ideal;

(b) B has exactly two non-maximal prime ideals.

Fields

C1. Let $\mathbb{F}_q$ be the finite field of q elements. Answer the following questions:

(a) List all subfields of $\mathbb{F}_{p^6}$ for a prime p. Justify your answer.

(b) Find a formula for the number of monic irreducible polynomials of degree 6 in $\mathbb{F}_p[X]$. Justify your answer.

C2. Let K/F be a quadratic extension of fields and M/F be a Galois extension over F containing K such that $\mathrm{Gal}(M/K)$ is a cyclic group of odd prime order p. Answer the following two questions:

(a) Determine the possible groups $\mathrm{Gal}(M/F)$ up to isomorphisms, and justify your answer.

(b) Find the number of intermediate fields L between F and M with $[L : F] = p$. Justify your answer.

C3. Find the degree of the splitting field E of $X^6 - 3$ over the following fields:

(a) $\mathbb{Q}[\sqrt{-3}]$ ($\mathbb{Q}$: the field of rational numbers);

(b) $\mathbb{F}_7$, the field with 7 elements;

(c) $\mathbb{F}_5$, the field with 5 elements,

and justify your answer.

Linear Algebra

D1. Let L be the subgroup of $\mathbb{Z}^3$ with generators $(3, 2, 1)$ and $(2, 2, 6)$. Represent the quotient group $A = \mathbb{Z}^3/L$ as a product of cyclic groups.

D2. List suitably the Jordan canonical forms of all matrices A that satisfy
(i) $tr(A) = 1$;
(ii) $tr(A)^2 = 2det(A) + tr(A^2)$.

D3. Let M be a matrix with complex entries. Deduce, using the structure theory of modules, that:
(i) $M = S + N$, where S is semisimple (i.e. diagonalizable), N is nilpotent, and N and S commute.
(ii) $N = P(M)$ where $P(X)$ is a polynomial with complex coefficients.