Algebra Qualifying Exam
Fall 2004

1. Groups

   (a) Is there a finite group G such that $G/Z(G)$ has 143 elements? ($Z(G)$ is the center of G.)

   (b) Prove that every group of order 30 has a subgroup of order 15.

   (c) Find all finite groups that have exactly three conjugacy classes.

2. Rings

   (a) Let X be a finite set and let A be the ring of all functions from X to the field R of real numbers. Prove that an ideal M of A is maximal if and only if there is an element $x \in X$ such that

   $$M = \{f \in A | f(x) = 0\}.$$

   (b) Describe all $n \in Z$ such that the ring Z/nZ has no idempotents other than 0 and 1.

   (c) A (non-commutative) ring R is called local if for every $a \in R$ either a or $1 - a$ is invertible. Prove that non-invertible elements of a local ring form a (two-sided) ideal.

3. Linear Algebra

   (a) Determine whether it is true in general that in $GL_n(\mathbf{C})$ every matrix is conjugate to its transpose.

   (b) Determine the number of conjugacy classes in $GL_3(\mathbf{C})$ whose elements A satisfy the polynomial $X^2 - 2X + 1 = 0$.

   (c) Let A be an n-by-n symmetric matrix with real entries and let, for $j \in [1,,n]$, A_j be the submatrix consisting of the entries of A in the first j rows and columns of A. Show A is positive definite iff $det(A_j) > 0$ for all $j \in [1,,n]$.

4. Fields

   (a) Let a be an integer and let p be a prime. Show that if a is not a p-th power, then $X^p - a$ is irreducible over Q.

(b) Show that if K and L are finite separable extensions of F with K Galois over F, such that $K \cap L = F$, then [KL:F]=[L:F][K:F]. Show that if neither K nor L are Galois over F, then this fact need not be true.

(c) By using several quadratic extensions of the rational function field in two variables $F = \mathbf{F}_2(X, Y)$ where $\mathbf{F}_2$ is the field with 2 elements, give an example of a field extension of finite degree of F that possesses infinitely many intermediate fields.