Algebra Qualifying Exam (Spring 2004)

Test Instructions: Everyone must do two problems in each of the four sections. If three problems of a section are tried, only the two problems of highest score count (the lowest score is ignored). On multiple part problems, do as many parts as you can; however, not all parts count equally.

GROUP THEORY

PROBLEM 1.

A group G is said to act transitively on a set S if for any element $s \in S$, then

$$S = Gs.$$

Suppose G is finite and that G acts transitively on S. Let $f(g)$ be the number of elements of S fixed by the action of $g \in G$ on S. Prove

$$|G| = \sum_{g \in G} f(g).$$

PROBLEM 2.

Classify all groups of order $2 \cdot 7 \cdot 11$.

PROBLEM 3.

Let G be a finite group and H a subgroup of G. Let $n = (G : H)$ be the index of H in G.

(a) Show that

$$(G : \cap_{x \in G} xHx^{-1})$$

is a factor of $n!$.

(b) Suppose that the index $(G : H)$ is the minimal prime factor of the order of G. Show H is a normal subgroup.

RING THEORY

PROBLEM 1.

Let R be a commutative noetherian ring with unity 1 and $f : R \to R$ a surjective ring homomorphism, i.e. $f(R) = R$. Show f is an isomorphism.

PROBLEM 2.

Let R be the ring $\mathbb{Q}[x]$ and let M be the submodule of R^2 generated by the elements $(1 - 2x, -x^2)$ and $(1 - x, x - x^2)$. According to the theory of modules over principal ideal domains, R^2/M is a direct sum of cyclic R modules of the form $R/P(x)$ for monic polynomials $P(x)$. Find such a direct sum decomposition explicitly in this case.

PROBLEM 3.

Suppose we are given a collection of polynomials in r variables with rational coefficients:

$$f_1, \dots f_N \in \mathbb{Q}[T_1, \dots T_r].$$

We define the complex algebraic set $V_{\mathbb{C}} \subset \mathbb{C}^r$ by

$$V_{\mathbb{C}} = \{(a_1, \dots a_r) \mid f_i(a_1, \dots a_r) = 0 \quad \text{for all } i \text{ from } 1 \text{ to } N\}.$$

Suppose $V_{\mathbb{C}}$ is not empty. Show that there is a finite extension K of $\mathbb{Q}$ and a point

$$(a_1, \dots a_r) \in V_{\mathbb{C}}$$

with all $a_k \in K$.

LINEAR ALGEBRA

PROBLEM 1.

(a) For which $z \in \mathbb{C}$ is

$$\begin{pmatrix} 1 & 2z \\ z-1 & 1 \end{pmatrix}$$

not similar over $\mathbb{C}$ to a diagonal matrix? Justify your answer.

(b) Let J_n be the $n \times n$ matrix each of whose entries is 1. Determine those $n \in \mathbb{Z}^+$ for which J_n is diagonalizable over $\mathbb{C}$ and give a diagonal matrix that is similar to J_n for such n.

PROBLEM 2.

Find an explicit formula for the determinant of a 3×3 complex matrix A as a polynomial in the traces $t_n = \mathrm{Tr}(A^n)$ for $n = 1, 2, \ldots$.

PROBLEM 3.

Let V be a vector space over $\mathbb{C}$ of dimension $d > 0$. Suppose that A, B, C are linear operators on V such that

$$AB - BA = C.$$

Suppose also that C commutes with both A and B. If V has no proper non-zero subspace that is left stable under all three operators, show that $d = 1$.

FIELD THEORY

PROBLEM 1.

Let K be a finite extension of $\mathbb{Q}$ obtained by adjoining to $\mathbb{Q}$ a root of $f(x) = x^6 + 3$.

(a) Show that K contains a primitive 6-th root of unity.

(b) Show that K is a Galois extension of $\mathbb{Q}$.

(c) Determine the number of fields F of degree 3 over $\mathbb{Q}$ with $F \subset K$.

PROBLEM 2.

Suppose that $f(x)$ is a polynomial in $\mathbb{Q}[x]$ of degree $d > 1$ with d roots $x_1, \ldots, x_d$ in $\mathbb{C}$. If $x_2 = ax_1$ for $a \in \mathbb{Q}$ different from -1, prove that $f(x)$ is reducible.

PROBLEM 3.

Let K be a field and L a finite extension of K. Consider the set A of all elements $x \in L$ with the property that $K[x]$ is a Galois extension of K with an abelian Galois group $\mathrm{Gal}(K[x]/K)$. Show that A is a subfield of L containing K .