

Algebra Qualifying Exam
Fall 2005

Test Instructions: All problems are worth 20 points. You are expected to do two problems in each of the four sections. Your total score will be computed by taking the two best scoring problems in each section. In problems where arguments must be given, you will lose points if you fail to state clearly the basic results you use.

(1) Groups

- (a) Let G be an abelian group generated by n elements. Prove that every subgroup of G can also be generated by n elements.
- (b) Let N be a normal subgroup of G . Prove that for a Sylow p -subgroup P of G , the intersection $P \cap N$ is a Sylow p -subgroup of N .
- (c) Is there a nontrivial action of the alternating group A_4 on a set of two elements?

(2) Rings

- (a) Let I and J be ideals of a commutative ring R with unit such that $I + J = R$. Prove that $I \cdot J = I \cap J$.
- (b) Prove that the factor ring $\mathbf{R}[x, y]/(y^2 - x^3)\mathbf{R}[x, y]$ is not a P.I.D.
- (c) Let x, y, z, t be elements of a (non-commutative) ring R such that $xz = yt = 1$, $xt = yz = 0$ and $zx + ty = 1$. Prove that the left R -modules R and $R \oplus R$ are isomorphic.

(3) Linear Algebra

- (a) Prove that if three distinct real numbers λ_i and three arbitrary numbers μ_j are given, then there exists a unique polynomial $f(x) \in \mathbf{R}[x]$ of degree at most 2 such that $f(\lambda_i) = \mu_i$.
- (b) Let K/F be a field extension of finite degree n and assume $K = F(\alpha)$ where α satisfies a polynomial $f(x)$ of degree n in $F[x]$. Let $\varphi : K \rightarrow K$ be the F -linear map $\varphi(x) = \alpha x$. Show that the eigenvalues of φ coincide with the roots of $f(x)$.
- (c) A bilinear form $A(x, y)$ on a vector space V over \mathbf{C} is called *alternating* if $A(v, w) = -A(w, v)$ for all $v, w \in V$ and it is called *non-degenerate* if, for each nonzero $v \in V$ there exists $w \in V$ such that $A(v, w) \neq 0$.
 - (i) Prove that any two non-degenerate alternating forms on $V = \mathbf{C}^2$ differ by a scalar multiple.
 - (ii) Does this remain true for $V = \mathbf{C}^n$ for $n > 2$? Either prove true or give a counterexample.

(4) Fields

- (a) Let \mathbf{F}_q be the finite field with $q = p^n$ elements and let $N : \mathbf{F}_q \rightarrow \mathbf{F}_p$ be the norm map, defined by

$$Nx = \prod \sigma(x)$$

where σ runs over the Galois group $G = \text{Gal}(\mathbf{F}_q/\mathbf{F}_p)$. Prove that N is surjective.

- (b) Let $\varphi(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ be an irreducible polynomial of degree 4 in $\mathbf{Q}[x]$ and let K be the field generated by the complex roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ of φ . Let F be the field generated by:

$$\begin{aligned}\beta_1 &= (\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4), \\ \beta_2 &= (\alpha_1 + \alpha_3)(\alpha_2 + \alpha_4), \\ \beta_3 &= (\alpha_1 + \alpha_4)(\alpha_2 + \alpha_3).\end{aligned}$$

Prove that K/F is an abelian extension, that is, the Galois group $H = \text{Gal}(K/F)$ is abelian. Hint: prove that the β_i are distinct and determine the possible elements of $\text{Gal}(K/\mathbf{Q})$ that fix them.

- (c) Let K be the splitting field of $\varphi(x) = x^{11} - 7$ over \mathbf{Q} . Describe the Galois group $G = \text{Gal}(K/\mathbf{Q})$ by giving generators and relations. Determine the number of quadratic subfields of K (a quadratic subfield is a subfield $E \subset K$ such that $[E : \mathbf{Q}] = 2$).