Algebra Qualifying Exam

Winter 2005

Test Instructions: Everyone must do two problems in each of the four sections. If three problems of a section are tried, only the two problems of highest score count (the lowest is ignored). For multiple part problems, do as many parts as you can; however, not all parts count equally.

Groups

A1. (a) If G is a simple group that has subgroup of index n, prove that the order of G is a factor of $n!$.

(b) Prove that there is no simple nonabelian group of order $p^e m$ with $e > 0$ for a prime $p > m$.

A2. An additive abelian group is called *divisible* if multiplication by n for every positive integer n is a surjective endomorphism.

(a) Show that if G is divisible, G/H is divisible for any subgroup H of G.

(b) Give an example (with a proof) of a divisible group for which the multiplication by n is not an automorphism for every positive integer n.

(c) Prove or disprove that there is only one isomorphism class of finitely generated divisible groups.

A3. Let G be a free abelian group of finite rank r.

(a) Show that there are only finitely many homomorphisms of G into $\mathbb{Z}/n\mathbb{Z}$ for each positive integer n.

(b) Find a formula of the number of surjective homomorphisms of G onto $\mathbb{Z}/p\mathbb{Z}$ for a prime p if $r = 2$.

Linear Algebra

B1. An $n \times n$ real symmetric matrix P is *positive definite* if the inner product $P(x, y) = {}^t x P y$ is positive definite (that is, $P(x, x) > 0$ for all $0 \neq x \in \mathbb{R}^n$). Let S be an $n \times n$ invertible real symmetric matrix. Let $W \subset \mathbb{R}^n$ be a subspace such that the inner product $S(x, y) = {}^t x S y$ is positive definite on W but S is not positive definite on $W + \mathbb{R}x$ for any $x \notin W$.

(a) Show that $\mathbb{R}^n = W \oplus W^\perp$ for $W^\perp = \{x \in \mathbb{R}^n | S(x, W) = 0\}$.

(b) For each $x \in \mathbb{R}^n$, writing $x = x_W \oplus x_{W^\perp}$ for $x_W \in W$ and $x_{W^\perp} \in W^\perp$, define $P(x, y) = S(x_W, y_W) - S(x_{W^\perp}, y_{W^\perp})$. Show that $PS^{-1} = SP^{-1}$ and P is positive definite.

(c) If P is symmetric positive definite and satisfies $PS^{-1} = SP^{-1}$, there exists a subspace W such that $P = S$ on W and $P = -S$ on $W^{\perp}$.

B2. Let V be a two dimensional vector space over a field F. Let $T : V \to V$ be a linear transformation of finite order m. Prove the following facts:
(a) If $F = \mathbb{Q}$, then $m \leq 6$.
(b) For any given positive integer N, there exists a finite field F and a nondiagonalizable T of order greater than N.

B3. Let V be a finite dimensional vector space over a field F and $T : V \to V$ be a linear transformation. Let $v \in V$ be a non-zero vector in V. Prove the following facts:
(a) There exists a monic polynomial $P(X)$ in $F[X]$ such that $P(T)v = 0$.
(b) Among monic polynomials $P(X) \in F[X]$ with $P(T)v = 0$, there exists a unique polynomial $P_0(X)$ of minimal degree.
(c) If $P(T)v = 0$, then $P_0(X)$ is a factor of $P(X)$ in $F[X]$.

Rings

C1. Let R be an integral domain. If $\mathfrak{m}$ is a maximal ideal in R, view the localization $R_{\mathfrak{m}} := S^{-1}R$, with $S = R \setminus \mathfrak{m}$, in the quotient field of R. Show that

$$R = \bigcap_{\mathfrak{m} \in \mathrm{Max}(R)} R_{\mathfrak{m}}.$$

C2. Let R be a commutative Artinian ring. Show that there are only finitely many prime ideals in R and every one of them is maximal.

C3. Let $R \subseteq A \subseteq B$ be commutative rings. Suppose that R is noetherian and B is a finitely generated R-algebra. Suppose that as an A-module B is finitely generated. Show that A is a finitely generated R-algebra.

Fields

D1. Show that the identity map is the only field automorphism of the real numbers. Show this is not true of the complex numbers.

D2. Let F be a field of positive characteristic p and f the polynomial $x^p - x - a \in F[x]$. Let K/F be a splitting field of f. Show that K/F is galois and determine explicitly (with proof) the Galois group of K/F.

D3. Let K/F be a finite extension of finite fields. Prove that the norm map $N_{K/F} : K \to F$ is surjective.