# Algebra Qualifying Exam

Fall 2006

**Test Instructions:** Everyone must do two problems in each of the four sections. If three problems of a section are tried, only the two problems of highest score count (the lowest is ignored). For multiple part problems, do as many parts as you can; however, not all parts count equally.

## Groups

G1. List all finite groups $G$ whose automorphism group has prime order. Justify your answer.

G2. Let $G$ be a finite group and $H$ be a non-normal subgroup of $G$ of index $n > 1$.

(a) Show that if $|H|$ is divisible by a prime $p \geq n$, then $H$ cannot be a simple group;

(b) Show that there is no simple group of order $504 = 2^3 \cdot 3^2 \cdot 7$. (Hint: Choose a good prime $\ell$, and let $G$ act on the set of Sylow $\ell$-subgroups getting an embedding of $G$ into a permutation group, and apply (a). You may use the fact that the alternating group $A_n$ is simple if $n \geq 5$.)

G3. Let

$$SL_2(\mathbb{Z}/p\mathbb{Z}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \middle| a, b, c, d \in \mathbb{Z}/p\mathbb{Z},\ ad - bc = 1 \right\},$$

where $p$ is an odd prime.

(a) Prove that any subgroup of a cyclic group is cyclic.

(b) Compute the order of $G$.

(c) Prove that for any odd prime $\ell$, the Sylow $\ell$-subgroup of $G$ is cyclic. (Hint: You may use the fact that the multiplicative group $F^\times$ of a finite field $F$ is cyclic).

## Rings

R1. Determine all prime ideals in the polynomial ring $\mathbb{Z}[x]$. Justify your work.

R2. Let $R$ be a noetherian domain. A non-zero element $x$ in $R$ is called a *prime element* if $(x)$ is a prime ideal. Prove all of the following:

(a) Every nonzero non-unit in $R$ is a product of irreducible elements.

(b) Every nonzero ideal $I \neq R$ in $R$ contains a (finite) product of non-zero prime ideals.

(c) If every nonzero prime ideal in $R$ contains a prime element then every irreducible element in $R$ is a prime element.

[You may not use theorems about UFD's]

R3. Let $R$ be a commutative ring and $M$ a finitely generated $R$-module. Suppose there exists a positive integer $n$ and a surjective $R$-module homomorphism $\varphi : M \to R^n$. Show that $\ker \varphi$ is also a finitely generated $R$-module.

## Fields

F1. Let $F$ be a finite field of positive characteristic $p$. Show that the unit group $F \setminus \{0\}$ of $F$ is a cyclic group and that $F$ is a Galois extension of $\mathbb{Z}/p\mathbb{Z}$.

F2. Let $f(x)$ be the polynomial $x^6 + 3$ over $\mathbb{Q}$ (the field of rational numbers). Determine the Galois group of $f(x)$, i.e., the Galois group of $K/\mathbb{Q}$ where $K$ is a splitting field of $f(x)$.

F3. Let $f(x)$ be an irreducible polynomial over $F$ and $K/F$ a normal extension. Show that $f(x)$ factors into irreducible polynomials over $K$ all of the same degree.

## Linear Algebra

L1. Let $V$ be an $n$-dimensional vector space over a field $F$ (for finite $n$) and $\sigma : F \to F$ be a field homomorphism (sending 1 to 1). We regard $V \otimes_{F,\sigma} F$ as an $F$-vector space via the $F$-multiplication given by $a(v \otimes \alpha) = (av) \otimes \alpha$ for $a, \alpha \in F$ and $v \in V$.

(a) Compute the formula of $\dim_F(V \otimes_{F,\sigma} F)$ if $F$ is a finite extension of a field $k$ fixed by $\sigma$, where $V \otimes_{F,\sigma} F$ is the tensor product over $F$ regarding $F$ as $F$-module via $\sigma$.

(b) Let $p$ be a prime. If $F = \mathbb{F}_p(x_1, \ldots, x_m)$ ($\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$) and $\sigma(\phi) = \phi^p$ for all $\phi \in F$, compute the formula of $\dim_F(V \otimes_{F,\sigma} F)$. Here $\mathbb{F}_p(x_1, \ldots, x_m)$ is the field of fractions of the polynomial ring $\mathbb{F}_p[x_1, \ldots, x_m]$ of $m$ variables.

(c) Give an example of a field $F$ of characteristic 3 and a homomorphism $\sigma : F \to F$ such that $\dim_F V \otimes_{F,\sigma} F = 2 \dim_F V$, and justify your example.

L2. Let $V$ be a finite dimensional vector space over the rational number field $\mathbb{Q}$ and $I : V \times V \to \mathbb{Q}$ be an alternating bilinear map (that is, $I(x, y) = -I(y, x)$ for all $x, y \in V$). We call $I$ *degenerate* if there exists nonzero $x \in V$ such that $I(x, V) = 0$, and $I$ is *non-degenerate* if $I$ is not degenerate.

(a) Show that if $V$ is two dimensional and $I$ is non-degenerate then there exists a basis $x, y$ such that $I(x, y) = 1$.

(b) Show that $I$ is degenerate if $\dim_{\mathbb{Q}} V$ is odd.

(c) If $\dim_{\mathbb{Q}} V = 2m$ is even and $I$ and $I'$ are two nondegenerate alternating forms on $V$, show that there exists an invertible linear transformation $T : V \to V$ such that $I'(x, y) = I(Tx, Ty)$ for all $x, y \in V$.

L3. Let $T : V \to V$ be a linear transformation on an $n$-dimensional vector space $V$ over $\mathbb{C}$ with $n \geq 1$. Suppose that $T^n = 0$ but $T^{n-1} \neq 0$.

(a) Compute $\dim \operatorname{Ker}(T^{101})$.

(b) If $S : V \to V$ is a linear transformation with $ST = TS$ and $\dim S(V) = n - 2$, compute $\dim S^{101}(V)$.