Algebra Qualifying Exam
Spring 2006

Test instructions: All problems are worth 20 points. You are expected to do two problems from each of the four sections. Your total score will be computed by taking the two best scoring problems from each section. In problems where an argument must be given, you will lose points if you fail to state clearly the basic results you use.

Groups

G1. Let G be a finite group. Let K be a normal subgroup of G and P a p-Sylow subgroup of K. Show that

$$G = KN_G(P).$$

G2.

(a) What is the order of $SL_2(F_4)$?

(b) Show there is an isomorphism from $SL_2(F_4)$ to A_5.
Hint: Consider the action of $SL_2(F_4)$ on the set of one dimensional subspaces of the vector space $(F_4)^2$ of dimension two over F_4.

G3. Let G be a group of order 2000 and suppose that P and P' are two distinct Sylow 5 subgroups of G. Let

$$I = P \cap P'.$$

(a) Prove that $|I| = 25$.

(b) Show that the index of $N_G(I)$ is at most 2.

Rings

R1. Suppose D is an integral domain and suppose that $D[x]$ is a principal ideal domain. Show D is a field.

R2. Let R be a commutative Noetherian ring with unit, and suppose M is a finitely generated R module. Suppose $f : M \to M$ is an R module homomorphism which is onto. Show that f is an isomorphism.

R3. Let R be a commutative ring with unit and m a maximal ideal of R.

(a) Suppose $I_1 \ldots I_n$ are ideals of R and that

$$m \supseteq I_1 \ldots I_n,$$

where $I_1 \ldots I_n$ is the product of the ideals. Show

$$m \supseteq I_k$$

for some k.

(b) Suppose that R satisfies the descending chain condition (dcc) on ideals, i.e. every strictly decreasing sequence of ideals is finite. Show R has only a finite number of maximal ideals. You may use part (a), but not theorems on the structure of rings satisfying the dcc.

Fields

F1.

(a) Show that the Galois group of the splitting field of

$$X^4 - 2$$

over $\mathbf{Q}$ has order 8.

(b) Is this Galois group isomorphic to the dihedral group, the quaternion group or one of the three abelian groups of order 8?

F2. Let F be a finite field.

(a) Show that more than half the elements of F are squares.

(b) Show that every element of F is the sum of two squares.

F3. Let K be a finite extension of the field F with no proper intermediate fields.

(a) If K/F is normal, show $[K : F]$ must be prime.

(b) Give an example to show that $[K : F]$ need not be prime if K/F is not normal, explaining why your example works.

Linear Algebra

L1. Let A be a 2×2 complex matrix and let W_A be the space of all 2×2 matrices that commute with A.

(a) What is the minimal possible dimension of W_A as A varies over all 2×2 complex matrices?

(b) Classify those A such that W_A has minimal dimension.

L2. Let A be a 3×3 matrix over a field F that satisfies

$$A^4 = I \quad \text{and} \quad A^2 \neq I,$$

where I is the identity matrix. Find all similarity classes of such A when

(a) $F = \mathbf{Q}$

(b) F is the field of two elements.

L3. Let $T_1, T_2, \ldots T_n$ be linear operators on a vector space of dimension m over a field F. Assume that

(a) $\dim \operatorname{im}(T_i) = 1$ for each i and

(b) $T_i^2 \neq 0$ and $T_i T_j = 0$ for $i \neq j$.

Show

$$n \leq m$$