Algebra Qualifying Exam
Spring 2007

Test Instructions: All problems are worth 20 points. You are expected to do two problems in each of the four sections. Your total score will be computed by taking the two best scoring problems in each section. In problems where arguments must be given, you will lose points if you fail to state clearly the basic results you use.

Groups

1. Let G be a simple group containing an element of order 21. Prove that every proper subgroup of G has index at least 10.

2. Find the number of subgroups of $\mathbb{Z}^n$ of index 5.

3. Let G be a group with cyclic automorphism group $\mathrm{Aut}(G)$. Prove that G is abelian.

Rings

1. Let D be a division ring (a ring with identity in which every non-zero element is invertible). Let $R = \mathrm{Mat}_n(D)$ be the ring of $n \times n$ matrices with entries from D. Prove that R has no two-sided ideals other than R itself and $\{0\}$.

2. Let $R = \mathrm{End}(V)$ be the ring of all linear endomorphisms of an infinite dimension complex vector space V with countable basis $\{e_1, e_2, \dots\}$. Prove that R and $R \oplus R$ are isomorphic as left R-modules.

3. (a) Give a description of all maximal ideals of the ring $\mathbf{C}[x, y]$. Justify your description. You may use the Nullstellensatz.

   (b) Let $M = (x^2 - y, y^2 - 5)$ be an ideal in $R = \mathbf{Q}[x, y]$. Prove that M is a maximal ideal.

Fields

1. Let $F = \mathbf{Q}(\zeta)$ where $\zeta = e^{2\pi i/5}$ and let E/F be a Cyclic Galois extension of degree 5. Prove that there exists $\alpha \in F$ such that $E = F(\sqrt[5]{\alpha})$. Hint: find $\alpha \in E$ such that $\sigma(\alpha) = \zeta\alpha$, where σ is a generator of the Galois group $\mathrm{Gal}(E/F)$.

2. Let $K = \mathbf{Q}(\sqrt{3}, \sqrt[7]{5})$.
   (a) Prove that K has only one subfield $F \subset K$ such that $[F : \mathbf{Q}] = 2$.
   (b) Find all subfields of K.
   (c) Find an element $u \in K$ such that $K = \mathbf{Q}(u)$.
   (d) Describe all elements $u \in K$ such that $K = \mathbf{Q}(u)$.

3. Let $\mathbf{F} = \mathbf{Z}/3$. First explain why $\mathbf{F}[x]/(x^2 - 2)$ is isomorphic to $\mathbf{F}[x]/(x^2 - 2x - 1)$. Then find an explicit isomorphism:
$$\phi : \mathbf{F}[x]/(x^2 - 2) \quad \to \quad \mathbf{F}[x]/(x^2 - 2x - 1).$$

Linear Algebra

1. Let A be a linear operator in a $\mathbb{Q}$-vector space V of dimension n such that the minimal polynomial of A has degree n. Prove that every linear operator on V that commutes with A is a polynomial in A over $\mathbb{Q}$.

2. Let $G = \mathrm{GL}_n(\mathbf{C})$ be the multiplicative group of invertible $n \times n$ matrices over $\mathbf{C}$. Prove that every element of finite order in G is conjugate to a diagonal matrix.

3. Let $A(x, y)$ be a bilinear form on a vector space V of finite dimension and
$$V_l = \{x \in V \quad \text{such that} \quad A(x,y) = 0 \quad \text{for all} \quad y \in V\},$$
$$V_r = \{y \in V \quad \text{such that} \quad A(x,y) = 0 \quad \text{for all} \quad x \in V\}.$$
Prove that $\dim V_l = \dim V_r$.