Algebra Qualifying Exam
Fall 2008

Test Instructions: All problems are worth 20 points. You are expected to do two problems in each of the four sections. Your total score will be computed by taking the two best scoring problems in each section. In problems where arguments must be given, you will lose points if you fail to state clearly the basic results you use.

Groups

G1. Let G be a finite group of order g and $\mathbb{Z}[G] \subset \mathbb{Q}[G]$ be the group algebras of G with integer and rational coefficients, respectively. Let

$$e\mathbb{Z}[G] = \{ea \in \mathbb{Q}[G] \mid a \in \mathbb{Z}[G]\}$$

for $e = g^{-1} \sum_{h \in G} h \in \mathbb{Q}[G]$, and define a group

$$G' = e\mathbb{Z}[G]/(\mathbb{Z}[G] \cap e\mathbb{Z}[G]).$$

Prove that G' is a group of order g. Find a necessary and sufficient condition to have $G' \cong G$ as groups, and justify your answer.

G2. Prove or disprove:

(a) the group $\mathrm{GL}_2(\mathbb{Q})$ of 2×2 matrices with rational coefficients has finite cyclic subgroups of order bigger than any given positive integer N.

(b) the group $\mathrm{GL}_2(\mathbb{R})$ of 2×2 matrices with real coefficients has finite cyclic subgroups of order bigger than any given positive integer N.

G3. Let G be an additive abelian group such that multiplication by n: $x \mapsto nx$ is surjective for all positive integers n. Let

$$G[n] = \{x \in G \mid nx = 0\}$$

and p be a prime.

(a) Prove that for a given integer $m \geq 1$, there are only finitely many subgroups H of order p^m in G if $G[p]$ is finite;

(b) Find a formula of the number of subgroups of G of order p if the order of $G[p]$ is p^3, and justify your answer.

Rings

R1. Let $A = M_n(F)$ be the ring of $n \times n$ matrices with entries in a field F.

(a) Prove that any left ideal of A is principal of the form Ax;

(b) How many left ideals of A if $n = 2$ and F is a finite field? (Give a simple formula of the number of maximal left ideals of A, and justify your answer.)

R2. Let A be a domain and $B = A[T, \frac{1}{T}]$ for an indeterminate T. Prove that the ring automorphism group $\mathrm{Aut}(B_{/A})$ of B inducing the identity on A is finite if and only if the group of invertible elements of A is finite.

R3. Consider the covariant functor $F : A \mapsto A^\times$ from the category ALG of commutative rings with identity to the category of sets. Here $A^\times$ is the group of invertible elements of A. Give an explicit form of a commutative ring R such that the functor F is isomorphic to the functor $A \mapsto \mathrm{Hom}_{\mathrm{ALG}}(R, A)$.

Fields

F1. Let L/F be a cubic (of degree 3) field extension of characteristic zero. Prove that there is an element $a \in F$ and a cubic field extension L_0 of the field $F_0 = \mathbb{Q}(a)$ such that L is the composite FL_0 of F and L_0 over F_0.

F2. A field extension L/F is said to be *balanced* if every field homomorphism $L \to L$ over F is an isomorphism.

(a) Prove that every algebraic (possibly infinite) field extension is balanced;

(b) Give an example of a balanced non-algebraic field extension.

F3. Let p be a prime integer and F a field such that the degree of every nontrivial finite field extension of F is divisible by p. Prove that for any finite field extension L/F, there exists a tower of field extensions $F = F_0 \subset F_1 \subset \cdots \subset F_n = L$ such that $[F_{i+1} : F_i] = p$ for any $i = 0, \ldots, n-1$.

Linear Algebra

L1. Let V be a finite dimensional vector space of dimension n over a field of characteristic 2. A bilinear form b on V is called *symmetric* (respectively, *alternating*) if $b(v, v') = b(v', v)$ for all $v, v' \in V$ (respectively, $b(v, v) = 0$ for all $v \in V$). Prove that the space $\mathrm{Alt}(V)$ of all alternating bilinear forms on V is a subspace of the space $\mathrm{Sym}(V)$ of all symmetric bilinear forms on V and find dimension of the factor space $\mathrm{Sym}(V)/\mathrm{Alt}(V)$.

L2. Find the number of conjugacy classes of elements of order 4 in the general linear group $\mathrm{GL}_4(\mathbb{Q})$.

L3. An $n \times n$ matrix A over a field F is called *regular over F* if the minimal and characteristic polynomials of A coincide. Prove that for a field extension L/F, an $n \times n$ matrix A over F is regular over F if and only if A is regular over L.