Algebra Qualifying Exam
Spring 2008

Test Instructions: Each problem is worth 20 points. You should do two problems in each section. In particular, *do not submit more than two problems in each section.* In each problem, you must explain clearly your reasoning and any theorems that you quote. Whether you pass or fail depends on your performance in each section, not only on the total score.

GROUPS

PROBLEM G1
Let p be a prime number. Show that a subgroup G of S_p which contains an element of order p and which contains a transposition must be the whole of S_p.

PROBLEM G2
Let $G = D_{2n}$ be the dihedral group of order $2n$ where $n \geq 3$. Prove that $\mathrm{Aut}(G)$ is isomorphic to the group of 2×2 matrices of the form

$$H = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} : \alpha \in (\mathbf{Z}/n)^*, \beta \in \mathbf{Z}/n \right\}.$$

PROBLEM G3
Let K be a normal subgroup of a finite group G. Let p be a prime. Let N_p be the number of p-Sylow subgroups (p-SSG) of G and N_p' the number in K.

(a) Show that $N_p = [G : N_G(P)]$ where P is any p-SSG of G and $N_G(P)$ is the normalizer of P in G.

(b) Prove that N_p' divides N_p.

RINGS

PROBLEM R1

Let D be an associative ring with unit having no zero divisors. Assume that the center of D contains a field k such that $\dim_k(D) < \infty$. Prove that D is a division algebra (i.e., every non-zero element is invertible).

PROBLEM R2

Let G be a finite group of order $|G| > 1$. The rational group ring $\mathbb{Q}[G]$ of G is the $\mathbb{Q}$-algebra consisting of all finite linear combinations

$$\sum_{g \in G} a_g \, g$$

where $a_g \in \mathbb{Q}$. Multiplication in $\mathbb{Q}[G]$ is defined by extending the group multiplication linearly.

(a) Show that $\mathbb{Q}[G]$ has a non-trivial idempotent: $\exists\, a \in \mathbb{Q}[G]$, $a \neq 0, 1$ with $a^2 = a$.
Hint: reduce to the case of a cyclic group $G = \langle x : x^n = 1 \rangle$.

(b) Show that $\mathbb{Q}[G]$ contains an invertible element u that is non-trivial, that is, not of the form $u = ag$ where $a \in \mathbb{Q}$ and $g \in G$.

(FYI: The Kadison-Kaplansky Conjecture claims that for G without torsion, the group algebra $\mathbb{Q}[G]$ contains no non-trivial idempotent. This conjecture is open in general.)

PROBLEM R3

Let R be a Noetherian ring and I any ideal of R. Prove that there exist prime ideals $P_1, ..., P_m$ of R such that

$$P_1 P_2 \cdots P_m \subset I$$

Hint: Show that if J is any non-prime ideal, then there exist $a, b \notin J$ such that $(J + a)(J + b) \subset J$. Then use the Noetherian property.

FIELDS

PROBLEM F1

Consider the polynomial $P(X) = X^5 - 4X + 2$ in $\mathbb{Q}[X]$.

(a) Show that P is irreducible and has 3 real roots and 2 complex ones.

(b) Show that the Galois group of P is S_5.

PROBLEM F2

Let $\zeta_n = \exp(2\pi i/n)$ be a primitive nth root of unity. Let $F_n = \mathbf{Q}(\zeta_n)$. Set

$$d_n = [F_n : \mathbf{Q}]$$

(a) Let $n = 6$. Find an irreducible polynomial of degree d_6 in $\mathbf{Q}[x]$ whose roots generate F_6.

(b) Let $n = 12$. Find an irreducible polynomial of degree d_{12} in $\mathbf{Q}[x]$ whose roots generate F_{12}.

PROBLEM F3

Let E/F be a finite, separable extension of fields. Prove that there exists $\alpha \in E$ such that $E = F(\alpha)$. State clearly any theorems you use in the proof.

LINEAR ALGEBRA

PROBLEM LA1

Let V and W be vector spaces over a field F and let V^* be the dual space of V. Let $\mathrm{Hom}(V, W)$ be the space of linear maps from V to W. There exists a natural linear map

$$T : V^* \otimes_F W \to \mathrm{Hom}(V, W)$$

defined by $T(f \otimes w)(v) = f(v) \cdot w$. Show that V is finite dimensional if and only if T is an isomorphism for all W.

PROBLEM LA2

Let $M_4(\mathbb{Q})$ be the ring of all 4×4 matrices with coefficients in $\mathbb{Q}$. Find a set of representatives for the conjugacy classes of elements $X \in M_4(\mathbb{Q})$ satisfying the equation $X^4 = 2X^2$.

PROBLEM LA3

Let V be a finite dimensional F-vector space and $T : V \to V$ a linear endomorphism. Show that there exists a decomposition

$$V = V_1 \oplus V_2$$

with the properties:

(1) $T(V_i) \subset V_i$ for $i = 1, 2$
(2) T is an isomorphism on V_1
(3) T is nilpotent on V_2.

Hint: Consider the sequences of subspaces $\mathrm{Im}(T) \supset \mathrm{Im}(T^2) \supset \cdots$ and that $\mathrm{Ker}(T) \subset \mathrm{Ker}(T^2) \subset \cdots$.