Algebra Qualifying Exam

Fall 2009

Test Instructions: Each problem is worth 20 points. Attempt at least 8 problems in any section. All tried problems will be graded.

Part 1: Categories and Functors.

(Cat 1). Let **Top** be the category of topological spaces. Recall that a morphism f in some category is called a monomorphism if, for any two morphisms g_1 and g_2 that can be precomposed with f, $fg_1 = fg_2$ implies $g_1 = g_2$. Dually, f is called an epimorphism if, for any g_1 and g_2 that can be post-composed with f, $g_1 f = g_2 f$ implies $g_1 = g_2$.

(a) Show that a continuous map $f : X \to Y$ is a monomorphism in **Top** if and only if f is one-to-one.

(b) Now show by example that an epimorphism in **Top** need not be onto.

(Cat 2). Let $F : \mathbf{Ab} \to \mathbf{Sets}$ be the forgetful functor from abelian groups to sets. Show that F does not have a right adjoint.

Part 2: Groups.

(Gr 1). Suppose A is an abelian group that is generated by n elements (or fewer). Show that any subgroup of A also can be generated by n elements (or fewer).

(Gr 2). Let $p < q$ be primes, $n \geq 0$ an integer and G a group of order pq^n. Show that G is solvable.

Part 3: Representations.

(Rep 1). Let G be a finite group and $\rho : G \to \mathrm{Gl}(V)$ a complex representation. Prove that (V, ρ) splits as a direct sum of irreducible representations of G. [**Note:** It does not suffice to just quote a theorem. You have to actually prove the statement.]

(Rep 2). Let G be a finite p-group and $\rho : G \to \mathrm{Gl}(V)$ a representation in a $\mathbb{F}_p$-vector space.

(a) Show that V has a one-dimensional G-invariant subspace W.

(b) Show by example that (V, ρ) need not split into a direct sum of irreducible representations.

Part 4: Commutative Rings.

(C1). Find a homomorphism $A \to B$ of commutative rings (sending the identity of A to the identity of B) and non-zero A-modules M, N such that the canonical map

$$B \otimes_A \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_B(B \otimes_A M, B \otimes_A N)$$

is the zero map, and justify your answer. Prove that the map is an isomorphism if M is a finitely generated projective A-module.

(C2). Prove the following facts:

(a) Any subring of $\mathbb{Q}$ sharing the identity with $\mathbb{Q}$ is a PID.

(b) For a subring $A \subset \mathbb{Z}[\sqrt{-1}]$ sharing the identity with $\mathbb{Z}[\sqrt{-1}]$, if $A \neq \mathbb{Z}$ and $A \neq \mathbb{Z}[\sqrt{-1}]$, A is not a PID.

Part 5: Non-commutative Rings.

(R1). Prove that every two-sided ideal of the ring $M_2(\mathbb{Z})$ is principal, i.e., generated by one element.

(R2). Let B be a central simple algebra over k of dimension 4 (so, the center of B is k and has no nontrivial two-sided ideals except for (0) and B itself). Prove the following facts.

(a) All left ideals of B have even dimensional.

(b) $B \cong M_2(k)$ if and only if B is not a division algebra, where $M_2(k)$ is the matrix algebra of 2×2 matrices with coefficients in k.

Part 6: Fields.

(F1). Prove that the multiplicative group $F \setminus \{0\}$ of a field F is a cyclic group if and only if F is a finite field.

(F2). Let $k = \mathbb{F}_2(t, s)$ be the field of fractions of two variable polynomial ring $\mathbb{F}_2[t, s]$, where $\mathbb{F}_2$ is the field with 2 elements. Write θ_a for a root of $T^2 + T + a = 0$ for $a \in k$ in an algebraic closure of k. An intermediate field M between K and k for a field extension K/k is a subfield in K containing k.

(a) How many intermediate fields between k and $k(\theta_t, \theta_s)$?

(b) How many intermediate fields between k and $k(\sqrt{t}, \sqrt{s})$?

Justify all your answers.