ALGEBRA QUALIFYING EXAM: Spring 2009

TEST INSTRUCTIONS All problems are worth 20 points. You are expected to do 2 problems from each of the four sections. Your total score will be computed by taking **the two best-scoring problems in each section**. In problems where arguments must be given, you will lose points if you fail to state clearly the basic results that you use.

GROUPS

G1. Let N be a normal subgroup of a finite group G and P a Sylow p-subgroup of G. Prove that $P \cap N$ is a Sylow p-subgroup of N.

G2. Let A be an abelian group generated by n elements. Prove that any subgroup of A can be generated by n elements.

G3. Let G be a finite group and $H \subset G$ a subgroup of index n. Suppose that $xH \cap Hy \neq \emptyset$ for any elements $x, y \in G \setminus H$. Prove that $|G| \geq n^2 - n$. (Hint: consider an action of G on $(G/H) \times (G/H)$.)

RINGS

R1. Show the the ring $\mathbf{Z}[2i]$ consisting of all complex numbers $a + 2bi$ with $a, b \in \mathbf{Z}$ is not a PID.

R2. Let $M_n(F)$ be a the matrix ring of $n \times n$ matrices over a field F. Suppose that there is a subring of $M_n(F)$ isomorphic to $M_m(F)$ for some m. Prove that m divides n.

R3. Two polynomials $f, g \in R[t]$ over a commutative ring R are called *coprime over R* if f and g generate the unit ideal in $R[t]$. Let $f, g \in \mathbf{Z}[t]$ be two polynomials such that f and g are coprime over $\mathbf{Q}$ and the residues of f and g in $(\mathbf{Z}/p\mathbf{Z})[t]$ are coprime for every prime integer p. Prove that f and g are coprime over $\mathbf{Z}$.

LINEAR ALGEBRA

LA1. A matrix N is said to be **nilpotent of order** k if $N^k = 0$, but $N^{k-1} \neq 0$. If N is nilpotent of order k, prove that

$$k = \min\{m \mid \ker(N^m) = \ker(N^{m+1})\}.$$

LA2. Consider the matrix

$$A = \begin{pmatrix} 0 & 2 & -2 \\ -2 & 0 & 1 \\ 2 & -1 & 0 \end{pmatrix}$$

(A) Prove that $(I - A)(I + A)^{-1}$ is an orthogonal matrix.
(B) Compute e^{At} as a function of t.

LA3. Let $\pi \in S_n$ be a permutation of n elements. Let P_π be the $n \times n$ matrix taking the standard basis vector $e_i \mapsto e_{\pi(i)}$ for all i. Describe the eigenvalues over $\mathbf{C}$ of P_π in terms of the cyclic decomposition of π.

FIELDS

F1. Let $p(x) \in K[x]$ be a monic irreducible polynomial of degree n whose discriminant $D \neq 0$, where K has characteristic $\neq 2$. Prove that the Galois group of p is contained in the alternating group A_n if and only in $\sqrt{D} \in K$.

F2. Let α be transcendental over $\mathbf{Q}$. What is the minimal polynomial of α over $\mathbf{Q}(\frac{\alpha^2+1}{\alpha-1})$?

F3. (A) Which roots of unity are contained in quadratic extensions of $\mathbf{Q}$, and which extensions are these?
(B) If $K/\mathbf{Q}$ is any field extension that contains a primitive n'th root of unity, and n is odd, then prove K contains a primitive $2n$'th root of unity.