Algebra Qualifying Exam
Fall 2010

Test Instructions: Each problem is worth 20 points. Attempt at least 8 problems. All tried problems will be graded.

Part 1: Categories and Functors

Problem 1 Let **Grp** be the category of groups and **Ab** the category of abelian groups. If $F\colon \mathbf{Ab} \to \mathbf{Grp}$ is the inclusion of categories, then find a left adjoint to F and prove that it is a left adjoint.

Problem 2 Let $F\colon \mathcal{C} \to \mathbf{Sets}$ be a covariant functor on a category $\mathcal{C}$. Assume that F is representable by an object $C_F \in \mathrm{Obj}(\mathcal{C})$. Identify which of the following statements are necessarily correct, proving your answers in each case.

(i) If $C \in \mathrm{Obj}(\mathcal{C})$ and $F(C) \neq \emptyset$, then there is an element $f \in \mathrm{Mor}_{\mathcal{C}}(C_F, C)$;

(ii) If G is a left adjoint of F, then G is representable;

(iii) If $C, D \in \mathrm{Obj}(\mathcal{C})$, and there is a map of sets $f\colon F(C) \to F(D)$, then there exists a $g \in \mathrm{Mor}_{\mathcal{C}}(C, D)$.

(iv) If $C, D \in \mathrm{Obj}(\mathcal{C})$, and there exists a $g \in \mathrm{Mor}_{\mathcal{C}}(C, D)$, then there exists a map of sets $f\colon F(C) \to F(D)$ (note we are not guaranteed in advance that $F(D) \neq \emptyset$).

(v) If $h \in \mathrm{Mor}_{\mathcal{C}}(D, C_F)$, then for any $C \in \mathrm{Obj}(\mathcal{C})$, there is a map of sets $F(C) \to \mathrm{Mor}_{\mathcal{C}}(D, C)$.

Part 2: Groups

Problem 3 Prove that there is no simple group of order 120.

Problem 4

1. Show from first principles (i.e. without using the classification theorem) that a subgroup of a finitely generated abelian group is finitely generated.

2. Let $M \subseteq \mathbf{Z}^3$ be the subgroup generated by elements $(13, 9, 2)$, $(29, 21, 5)$, and $(2, 2, 2)$. Determine the isomorphism class of the quotient group $\mathbf{Z}^3/M$.

Part 3: Representations

Problem 5 Prove that if a finite group G acts transitively on a set S having more than one element then there exists an element of G which fixes no element of S. (Hint: first prove a formula that counts the average number of fixpoints of an arbitrary (i.e. not necessarily transitive) action.)

Problem 6 Let R be the 5 dimensional tautological representation of S_5. Show that R is isomorphic to the direct sum of the trivial representation and an irreducible 4 dimensional representation. State clearly any principles or theorems that you use.

Part 4: Commutative Rings and Modules

Problem 7 Let k be a field and let f be an irreducible element of the polynomial ring $k[x, y]$.
(i) Describe the localization $k[x, y]_{\mathcal{P}}$ where $\mathcal{P} = (f)$. Prove that it is a subring of the field of rational functions $k(x, y)$.
(ii) For any $r \in k(x, y)$, prove that

$$r \in \mathrm{Im}(k[x, y]_{\mathcal{P}} \to k(x, y))$$

for all but finitely many choices of f.

Problem 8 Let k be an algebraically closed field, $R = k[x_1, \ldots, x_n]$ and $f: R \to R^d$ be given by $f(p) = (pf_1, \ldots, pf_d)$ with all $f_i \in R$. Let $\mathcal{M}$ be the ideal generated by $x_1 - a_1, \ldots, x_n - a_n$, where $a_i \in k$ for all i. Consider for an integer r, $r \geq 1$, the map

$$f_{\mathcal{M}^r}: R \otimes_R (R/\mathcal{M}^r) \to R^d \otimes_R (R/\mathcal{M}^r)$$

induced by f. Prove that $\ker(f_{\mathcal{M}^r}) \neq 0$ if and only if $f_j(a_1, \ldots, a_n) = 0$ for all j.

Part 5: Non-Commutative Rings

Problem 9 Let R be a ring with 1. Show that the only two-sided ideals of $M_{n\times n}(R)$, the $n \times n$ matrices with entries in R, are of the form $M_{n\times n}(I)$ for some 2-sided ideal I in R.

Problem 10 Let G be a group and $\mathbf{Z}_G$ the integral group ring, i.e. the set of formal finite sums $\sum_i n_i(g_i)$ where $n_i \in \mathbf{Z}$ and $g_i \in G$ for all i. Multiplication $f \cdot h$ of two elements $f = \sum_i n_i(g_i)$ and $h = \sum_j m_j(g_j)$ is

$$f \cdot h = \sum_{i,j} n_i m_j (g_i g_j).$$

Let I be the two-sided ideal

$$I = \{\sum_i n_i(g_i) \mid \sum_i n_i = 0\}.$$

Construct a natural map

$$F\colon I/I^2 \to G/[G,G]$$

and prove that this map is an isomorphism of $\mathbf{Z}$-modules.

Part 6: Fields

Problem 11 Show that the extension of $\mathbf{Q}$ generated by $5^{1/2} + 2^{1/3}$ is equal to $\mathbf{Q}[5^{1/2}, 3^{1/3}]$.

Problem 12 Show that the multiplicative group of a finite field is cyclic and use this result to prove that the polynomial $X^4 + 1$ is never irreducible over any finite field.