Algebra Qualifying Exam
Spring 2010

Test Instructions: Each problem is worth 20 points. Attempt at least 8 problems. All tried problems will be graded.

Part 1: Categories and Functors

Problem 1 Show that the functor from (unitary) rings to groups sending a ring A to its group of units $A^\times$ is co-representable by a ring R. In other words, show there exists a ring R and a natural isomorphism of functors $Hom_{rings}(R, A) \to A^\times$.

Problem 2 (i) Define what is means for two categories to be equivalent. (ii) A groupoid $\mathcal{G}$ is a category such that all morphisms are isomorphisms. $\mathcal{G}$ is called connected if for any two objects x and y, $Hom_{\mathcal{G}}(x, y)$ is non-empty. Show that any non-empty connected groupoid is equivalent to a group, that is, a groupoid with one object.

Part 2: Groups

Problem 3 Determine, using the structure theory of abelian groups or otherwise, all finitely generated abelian groups A such that the group $Aut(A)$ of automorphisms of A is finite. State clearly any basic theorems that you use. Determine the order of the automorphism group of $A = \mathbf{Z} \times \mathbf{Z}/4\mathbf{Z}$.

Problem 4 Show that if S_4 denotes the symmetric group of degree 4, and σ is an automorphism of S_4, and $\tau \in S_4$ is a transposition, then $\sigma(\tau)$ is also a transposition. By studying the action of σ on transpositions, show that every automorphism of S_4 is inner. (Remark: this result holds for all S_n except n=6.)

Part 3: Representations

Problem 5 Give the total number and the dimensions of the irreducible complex representations of S_4. Prove your answer.

Problem 6 State and prove Schur's Lemma which describes $Hom_G(V, W)$ for V and W finite dimensional irreducible complex representations of a finite group.

Part 4: Commutative Rings and Modules

Problem 7 Show that the group of units of the ring $\mathbf{Z}/N\mathbf{Z}$ is cyclic iff N is either a power of an odd prime number, twice a power of an odd prime number, or 4.

Problem 8 Let F be the field with 2 elements and let $R = F[X]$. List up to isomorphism all R-modules with 8 elements that are cyclic.

Part 5: Non-Commutative Rings

Problem 9 Let A be a left Noetherian ring. Show that every left invertible element $a \in A$ is two-sided invertible.

Problem 10 Let F be a field and V a finite-dimensional F-vector space. Show that $R = End_F(V)$ has no non-trivial two-sided ideals.

Part 6: Fields

Problem 11 Let F be a field of characteristic zero containing the p-th roots of unity for p a prime. Show that the cyclic extensions of degree p of F in any algebraic closure $\overline{F}$ of F are in one to one correspondence with the subgroups of order p of $F^*/(F^*)^p$.

Problem 12 Determine all fields F such that the multiplicative group of F is finitely generated.