ALGEBRA QUALIFYING EXAM
FALL 2011

Do all the following 10 problems (see reverse). Good luck!

Problem 1. For a finite field $\mathbb{F}$, prove that the order of the group $\mathrm{SL}_2(\mathbb{F})$, of 2×2 matrices with determinant 1, is divisible by 6.

Problem 2. Let G be a non-trivial finite group and p a prime. If every subgroup $H \neq G$ has index divisible by p, prove that the center of G has order divisible by p.

Problem 3. Let R be a local UFD of Krull dimension 2 (meaning that the maximal integer m for which there exist strict inclusions of prime ideals $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_m$ in R is exactly 2). Let $\pi \in R$ be neither zero nor a unit. Prove that $R[\frac{1}{\pi}]$ is a PID.

Problem 4. Let p be a prime. Prove that the nilradical of the ring $\mathbb{F}_p[X] \otimes_{\mathbb{F}_p[X^p]} \mathbb{F}_p[X]$ is a principal ideal.

Problem 5. Let $\mathbb{F}$ be a finite field and $\overline{\mathbb{F}}$ be an algebraic closure of $\mathbb{F}$. Let K be a subfield of $\overline{\mathbb{F}}$ generated by all roots of unity over $\mathbb{F}$. Show that any simple K-algebra of finite dimension over K is isomorphic to the matrix algebra $M_n(K)$ for a positive integer n.

Problem 6. Let R be a commutative ring and let M be a finitely generated R-module. Let $f : M \to M$ be R-linear such that $f \otimes \mathrm{id} : M \otimes_R R[T] \to M \otimes_R R[T]$ is surjective. Prove that f is an isomorphism.

Problem 7. Let $\mathcal{C}$ be the category of semi-symplectic topological quantum paramonoids of Rice-Paddy type, satisfying the Mussolini-Rostropovich equations at infinity. Let X, Y be objects of $\mathcal{C}$ such that the functors $\mathrm{Mor}_{\mathcal{C}}(X, -)$ and $\mathrm{Mor}_{\mathcal{C}}(Y, -)$ are isomorphic, as covariant functors from $\mathcal{C}$ to sets. Show that X and Y are isomorphic in $\mathcal{C}$.

Problem 8. Let Γ be the Galois group of the polynomial $X^5 - 9X + 3$ over $\mathbb{Q}$. Determine Γ. [Hint: Show that Γ contains an element of order 5 and that Γ contains a transposition, in a sense to be made precise.]

Problem 9. (We denote by $\mathbb{F}G$ the group algebra of G.)

(a) Is there a group G with $\mathbb{C}G$ isomorphic to $\mathbb{C} \times \mathbb{C} \times \mathrm{M}_2(\mathbb{C})$?

(b) Is there a group G with $\mathbb{Q}G$ isomorphic to $\mathbb{Q} \times \mathbb{Q} \times \mathrm{M}_3(\mathbb{Q})$?

Problem 10. Let K/k be an extension of finite fields. Show that the norm $N_{K/k} : K \to k$ is surjective.