Algebra Qualifying Exam

1. Let G be a group of order n. Show that there are two subgroups H_1 and H_2 of the symmetric group S_n, both isomorphic to G such that $h_1h_2 = h_2h_1$ for all $h_1 \in H_1$ and $h_2 \in H_2$.

2. Let G be a finitely generated group. Show that G contains only finitely many subgroups of any fixed finite index.

3. Let R be a noetherian domain. Show that every nonzero nonunit in R is a product of irreducible elements and that R is a UFD if and only if every nonzero nonunit in R is a product of prime elements, where an element is prime if it generates a nonzero prime ideal.

4. Prove that every prime ideal in $\mathbf{Z}[t]$ can be generated by two elements.

5. Let F be a field having no nontrivial field extensions of odd degree and K/F a finite field extension. Show if K has no field extensions of degree two, then F is perfect and K is algebraically closed.

6. Prove that over a finite field there are irreducible polynomials of any positive degree.

7. Let $T : V \to V$ be a linear operator on a finite dimensional vector space. Prove that the characteristic polynomial P_T is irreducible if and only if T has no nontrivial invariant subspaces.

8. Let V be a finite dimensional vector space over a field F. Prove that every right ideal in $\mathrm{End}_F(V)$ is of the form $\{T : \mathrm{im}(T) \subset W\}$ for a unique subspace W of V.

9. Let G be a finite group of invertible linear operators on a finite dimensional vector space V over the field of complex numbers. Prove that if $(\dim V)^2 > |G|$, then there is a proper non-zero subspace $W \subset V$ such that $g(w) \in W$ for every $g \in G$ and every $w \in W$.

10. Show that there is a (covariant) functor from the category of groups to the category of sets taking a group G to the set of all subgroups of G. Determine whether this functor is representable.