Algebra Qualifying Exam, Fall 2012

(1) Let p be a prime integer and let G be a (finite) p-group. Write C for the subgroup of central elements $x \in G$ satisfying $x^p = 1$. Let N be a normal subgroup of G such that $N \cap C = \{1\}$. Prove that $N = \{1\}$.

(2) Let A be an $n \times n$ matrix over a field F having only one invariant factor. Prove that every $n \times n$ matrix over F that commutes with A is a polynomial in A with coefficients in F.

(3) Let F be a field and let n be a positive integer such that F has no nontrivial field extensions of degree less than n. Let $L = F(x)$ be a field extension with $x^n \in F$. Prove that every element in L is a product of elements of the form $ax + b$ with $a, b \in F$.

(4) Let F be the functor from the category of rings to the category of sets taking a ring R to the set $\{x^2 \mid x \in R\}$. Determine whether F is representable.

(5) Let G and H be finite groups and let V and W be irreducible (over $\mathbb{C}$) G- and H-modules respectively. Prove that the $G \times H$-module $V \otimes W$ is also irreducible.

(6) Let D_n be a dihedral group of order $2n > 4$; so, it contains a cyclic subgroup C of order n on which $\sigma \in D_n$ outside C acts as $\sigma c \sigma^{-1} = c^{-1}$ for all $c \in C$. When is the cyclic subgroup C with the above property unique? Determine all n for which D_n has unique cyclic subgroup C and justify your answer.

(7) Let D be a central simple division algebra of dimension 4 over a field F. If a quadratic extension K/F can be isomorphically embedded into D as F-algebras, prove that $D \otimes_F K$ is isomorphic to the 2×2 matrix algebra $M_2(K)$ as K-algebras.

(8) How many monic irreducible polynomials over $\mathbb{F}_p$ of prime degree l are there? Here $\mathbb{F}_p$ is the field of p elements for a prime number p. Justify your answer.

(9) Consider a covariant functor $F : R \mapsto R^\times$ from the category of commutative rings with a multiplicative identity into the category of sets. Let $G = \mathrm{Aut}_{functors}(F)$ be made up of natural transformations $I : F \longrightarrow F$ having an inverse $J : F \longrightarrow F \in G$ such that $I \circ J = J \circ I$ is the identity natural transformation. Prove first that F is representable by a ring, that G is a finite set, and find the order of the group G with justification.

(10) For a finite field $\mathbb{F}$ of order q, consider the polynomial ring $R = \mathbb{F}[x]$, and let L be a free R-module of rank 2. Give the number of R-submodules M such that $xL \subsetneq M \subsetneq L$, and justify your answer.