Algebra Qualifying Exam, Spring 2012

1. Let V be a finite dimensional space over $\mathbb{Q}$ and $G \subset GL(V)$ a finite subgroup. Prove that the $\mathbb{Q}$-subalgebra of $End(V)$ generated by G is semisimple.

2. Let V be the vector space of all $a \in \mathbb{R}^n$ such that $a_1 + a_2 + \cdots + a_n = 0$. The symmetric group S_n acts naturally on V. Prove that the S_n-module V is simple.

3. Let R be a commutative local ring, and P a finitely generated projective R-module. Show that P is a free module.

4. Let R be the subring of $M_3(\mathbb{R})$ consisting of all matrices (a_{ij}) with $a_{31} = a_{32} = 0$. Determine the Jacobson radical of R.

5. Let G be a finite group, K a normal subgroup and P a p-Sylow subgroup of G. Prove that $P \cap K$ is a p-Sylow subgroup of K.

6. Determine the Galois group of the polynomial $X^8 + 16$ over $\mathbb{Q}$.

7. Let F be a finite field with p^n elements (p is prime). Find the number of elements of F that can be written in the form $a^p - a$ for some $a \in F$.

8. Let R be a reduced (meaning, no non-zero nilpotent elements) commutative ring that has a unique proper prime ideal. Show that R is a field.

9. Let R be a flat commutative $\mathbb{Z}$-algebra, and $\mathbf{Mod}_R$ the category of R-modules. Suppose that I is an injective R-module. Show that the underlying abelian group of I is divisible.

10. Determine the automorphism group of the symmetric group S_3 up to isomorphism.