Algebra Qualifying Exam, Fall 2013

Do the following ten problems.

1. How many groups are there up to isomorphism of order pq where $p > q$ are prime integers?

2. Show that there are up to isomorphism exactly two nonabelian groups G of order 8. Prove that each of them has an irreducible complex representation of dimension 2.

3. For a positive integer n, let $\Phi_n(X)$ be the nth cyclotomic polynomial. If a is an integer and p a prime not dividing n, such that p divides $\Phi_n(a)$, show that the order of a mod p is n. Using this prove that there are infinitely many primes p such that p is 1 modulo n.

4. Given a field K of characteristic p, when is an α that is algebraic over K said to be separable? Show that if α is algebraic over K, then α is separable if and only if $K(\alpha) = K(\alpha^{p^n})$ for all positive integers n.

5. Let G be a finite group which has the property that for any element $g \in G$ of order n, and an integer r prime to n, the elements g and g^r lie in the same conjugacy class. Then show that the character of every representation of G takes values in the rational numbers $\mathbb{Q}$ (in fact even the integers $\mathbb{Z}$). (Hint: Use Galois theory.)

6. Let I be an ideal of a commutative ring and $a \in R$. Suppose the ideals $I + Ra$ and $(I : a) := \{x \in R \mid ax \in I\}$ are finitely generated. Prove that I is also finitely generated.

7. Give an example of a 10×10 matrix over $\mathbb{R}$ with minimal polynomial $(X + 1)^2(X^4 + 1)$ which is not similar to a matrix with rational coefficients.

8. Suppose that E/F is an algebraic extension of fields such that every nonconstant polynomial in $F[X]$ has at least one root in E. Show that E is algebraically closed.

9. Prove that is $ab = 1$ in a semisimple ring, then $ba = 1$.

10. Let A be the functor from the category of groups to the category of (unital) rings taking a group G to the group ring $\mathbb{Z}[G]$ of all finite formal sums $\sum_{g \in G} a_g g$ with $a_g \in \mathbb{Z}$ (the product in $\mathbb{Z}[G]$ is induced by the group operation in G). Prove that A has a right adjoint functor.