Algebra Qualifying Exam, Spring 2013

1: Let G be a free abelian group of rank r, so G is isomorphic to $\mathbb{Z}^r$ as groups. Show that G has only finitely many subgroups of a given finite index n.

2: Assume that L is a Galois extension of the field of rational numbers $\mathbb{Q}$ and that $K \subset L$ is the subfield generated by all roots of unity in L. Suppose that $L = \mathbb{Q}[a]$, where $a^n \in \mathbb{Q}$ for some positive integer n. Show that the Galois group $\mathrm{Gal}(L/K)$ is cyclic.

3: Let $K \subset L$ be an algebraic extension of fields. An element a of L is called abelian if $K[a]$ is a Galois extension of K with abelian Galois group $\mathrm{Gal}(K[a]/K)$. Show that the set of abelian elements of L is a subfield of L containing K.

4: Let $\mathbb{F}_2$ be the field with 2 elements and let $R = \mathbb{F}_2[x]$. List, up to isomorphism, all R–modules with 8 elements.

5: Let R be a commutative local ring, so R has a unique maximal ideal M.
a) Show that if $x \in M$ then $1 - x$ is invertible.
b) Show that if R is Noetherian and if I is an ideal such that $I^2 = I$ then $I = 0$.

6: Let D be a division ring of characteristic 0. Assume that D has dimension 2 as a $\mathbb{Q}$-vector space. Show that D is commutative.

7: Let $F = \mathbb{F}_2$ be the field with two elements. Show that there is a ring homomorphism $F[GL_2(F)] \to M_2(F)$ that sends the element g in the group ring to the matrix $g \in M_2(F)$. Show that this homomorphism is surjective. Let K be the kernel; since it is a left ideal, it is a (left) $GL_2(F)$-module. Is this module indecomposable? (Reminder: a module is indecomposable if it is not the direct sum of two proper submodules.) Describe the simple modules in its composition series.

8: Let $\mathbf{C}$ and $\mathbf{D}$ be additive categories, and let $\Phi : \mathbf{C} \to \mathbf{D}$ be a functor. Show: If Φ has a right adjoint, then Φ commutes with direct sums and for any two objects x and y in $\mathbf{C}$, the map $\Phi_{x,y} : \mathrm{Hom}_{\mathbf{C}}(x, y) \to \mathrm{Hom}_{\mathbf{D}}(x, y)$ is a homomorphism.

9: Let D be an associative ring without zero divisors, and assume the center of D is a field over which D is a finite-dimensional vector space. Prove that D is a division algebra.

10: Let G be a finite group of order n and $\rho : G \to GL(V)$ a complex representation of G of dimension n. Show that ρ cannot be irreducible.