Algebra Qualifying Exam, Fall 2014

Please do the following ten problems. Write your UID number ONLY, not your name.

(1) Let G be a finite group. Let $\mathbb{Z}[G]$ be the group algebra of G with augmentation ideal $\mathfrak{a}$. Show that $\mathfrak{a}/\mathfrak{a}^2 \cong G/G'$ as abelian groups for the derived group G' of G.

(2) Let $\mathbb{F}_p$ denote the finite field of p elements. Consider the covariant functor F from the category of commutative $\mathbb{F}_p$-algebras with a multiplicative identity to abelian groups sending a ring R to its p-th roots of unity, that is, $F(R) = \{\zeta \in R | \zeta^p = 1\}$. Answer the following questions and justify your answers.

(a) Give an example of a finite local ring R such that $F(R)$ has p^2 elements.

(b) Let $\mathrm{Aut}(F)$ be the set of natural transformations of F into itself inducing a group automorphism of $F(A)$ for all commutative rings A with identity. Prove that F is representable and use the Yoneda Lemma to compute the order of $\mathrm{Aut}(F)$.

(3) Pick a non-zero rational number x. Determine all possibilities for the Galois group G of the normal closure of $\mathbb{Q}[\sqrt[4]{x}]$ over $\mathbb{Q}$, where $\sqrt[4]{x}$ is the root of $X^4 - x$ with maximal degree over $\mathbb{Q}$.

(4) Let D be a 9-dimensional central division algebra over $\mathbb{Q}$ and $K \subset D$ be a field extension of $\mathbb{Q}$ of degree > 1. Show that $K \otimes_{\mathbb{Q}} K$ is not a field and deduce that $D \otimes_{\mathbb{Q}} K$ is no longer a division algebra.

(5) Let R be a commutative algebra over $\mathbb{Q}$ of finite dimension n. Let $\rho : R \to M_n(\mathbb{Q})$ be the regular representation, and define $\mathrm{Tr} : R \to \mathbb{Q}$ by the matrix trace of ρ. If the pairing $(x, y) = \mathrm{Tr}(xy)$ is non-degenerate on R, prove that R is semi-simple.

(6) Let G be a finite group and let p be the smallest prime number dividing the order of G. Assume G has a normal subgroup H of order p. Show that H is contained in the center of G.

(7) Let G be a finite group and P a Sylow 2-subgroup of G. Assume P is cyclic, generated by an element x. Show that the signature of the permutation of G given by $g \mapsto xg$ is -1. Deduce that G has a non-trivial quotient of order 2.

(8) Let A be a ring. Assume there is an infinite chain of left ideals $I_0 \subset I_1 \subset \cdots \subset A$ with $I_i \neq I_{i+1}$ for $i \geq 0$. Show that A has a left ideal that is not finitely generated as a left A-module.

(9) Let A be a ring and let $i, j \in A$ such that $i^2 = i$ and $j^2 = j$. Show that the left A-modules Ai and Aj are isomorphic if and only if there are $a, b \in A$ such that $i = ab$ and $j = ba$.

(10) Let n be a positive integer. Let A_n be the $\mathbb{Q}$-algebra generated by elements $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ with relations

$$x_i x_j = x_j x_i,\ y_i y_j = y_j y_i \text{ and } y_i x_j - x_j y_i = \delta_{ij} \text{ for } 1 \leq i, j \leq n.$$

Show that there is a representation of A_n on the vector space $\mathbb{Q}[t_1, \ldots, t_n]$ where x_i acts by multiplication by t_i and y_i acts as $\frac{\partial}{\partial t_i}$.