Algebra Qualifying Exam
Spring 2014

Please do the following ten problems. Write your UID number ONLY, not your name.

(1.) Let $L : \mathbf{C} \to \mathbf{D}$ be a functor, left adjoint to $R : \mathbf{D} \to \mathbf{C}$. Show: if the counit $L \circ R \to id_{\mathbf{D}}$ is a natural isomorphism, then R is fully faithful.

(2.) Let A be a central division algebra (of finite dimension) over a field k. Let $[A, A]$ be the k-subspace of A spanned by the elements $ab - ba$ with $a, b \in A$. Show that $[A, A] \neq A$.

(3.) Given $\phi : A \to B$ a surjective morphism of rings, show that the image by ϕ of the Jacobson radical of A is contained in the Jacobson radical of B.

(4.) Let G be a group and H a normal subgroup of G. Let k be a field and let V be an irreducible representation of G over k. Show that the restriction of V to H is semisimple.

(5.) Let G be a finite group acting transitively on a finite set X. Let $x \in X$ and let P be a Sylow p-subgroup of the stabilizer of x in G. Show that $N_G(P)$ acts transitively on X^P.

(6.) Let A be a ring and M a noetherian A-module. Show that any surjective morphism of A-modules $M \to M$ is an isomorphism.

(7.) Let G be a finite group and let $s, t \in G$ be two distinct elements of order 2. Show that the subgroup of G generated by s and t is a dihedral group. (Recall that the dihedral groups are the groups $D(m) = \langle g, h | g^2 = h^2 = (gh)^m = 1 \rangle$ for some $m \geq 2$).

(8.) Let F be a finite field. Without using any of the theorems on finite fields, show that F has a field extension of degree 2.

(9.) Let G be a finite group. Show that there exist fields $F \subset E$ such that E/F is Galois with group G.

(10.) Let F be a field. Show that the polynomial ring $F[t]$ has infinitely many prime ideals. Also prove that algebraically closed fields are infinite.