Algebra Qualifying Exam, Fall 2015

Please do the following ten problems. Write your UID number ONLY, not your name.

(1) Show that the inclusion map $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epimorphism in the category of rings with multiplicative identity.

(2) Let R be a principal ideal domain with field of fractions K.

(a) Let S be a non-empty multiplicatively closed subset of $R \setminus \{0\}$. Show that $R[S^{-1}]$ is a principal ideal domain.

(b) Show that any subring of K containing R is of the form $R[S^{-1}]$ for some multiplicatively closed subset S of $R \setminus \{0\}$.

(3) Let k be a field and define $A = k[X, Y]/(X^2, XY, Y^2)$.

(a) What are the principal ideals of A?

(b) What are the ideals of A?

(4) Let K be a field and let L be the field $K(X)$ of rational functions over K.

(a) Show that there are two unique K-automorphisms f and g of the field $L = K(X)$ such that $f(X) = X^{-1}$ and $g(X) = 1 - X$. Let G be the subgroup of the group of K-automorphisms of L generated by f and g. Show that $|G| > 3$.

(b) Let $E = L^G$. Show that

$$P = \frac{(X^2 - X + 1)^3}{X^2(X-1)^2} \in E.$$

(c) Show that $L/K(P)$ is a finite extension of degree 6.

(d) Deduce that $E = K(P)$ and that G is isomorphic to the symmetric group S_3.

(5)

(a) Let G be a group of order $p^e v$ with v and e positive integers, p prime, $p > v$, and v not a multiple of p. Show that G has a normal Sylow p-subgroup.

(b) Show that a nontrivial finite p-group has nontrivial center.

(6) Let F be a field of characteristic not 2. Let a and b be nonzero elements of F. Let R be the F-algebra

$$R = F\langle i, j\rangle/(i^2 - a, j^2 - b, ij + ji),$$

the quotient of the free associative algebra on 2 generators by the given two-sided ideal.

(a) Let $\overline{F}$ be an algebraic closure of F. Show that $R \otimes_F \overline{F}$ is isomorphic as an $\overline{F}$-algebra to the matrix algebra $M_2(\overline{F})$.

(b) Give a basis for R as an F-vector space, justifying your answer. (You may use (a).)

(7) Show that the symmetric group S_4 has exactly two isomorphism classes of irreducible complex representations of dimension 3. Compute the characters of these two representations.

(8) Let F be a field. Show that the group $SL(2, F)$ is generated by the matrices $\begin{pmatrix} 1 & e \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ e & 1 \end{pmatrix}$ for elements e in F.

(9)

(a) Let R be a finite-dimensional associative algebra over a field F. Show that every element of the Jacobson radical of R is nilpotent.

(b) Let R be a ring. Is an element of the Jacobson radical of R always nilpotent? Is a nilpotent element of R always in the Jacobson radical? Justify your answers.

(10) Let p be a prime number. For each abelian group K of order p^2, how many subgroups H of $\mathbf{Z}^3$ are there with $\mathbf{Z}^3/H$ isomorphic to K?