Algebra Qualifying Exam, Spring 2015

Please do the following ten problems. Write your UID number ONLY, not your name.

(1) What are the coproducts in the category of groups?

(2) Let $\mathcal{C}$ be the category of groups and $\mathcal{C}'$ be its full subcategory with objects the abelian groups. Let $F : \mathcal{C}' \to \mathcal{C}$ be the inclusion functor. Determine the left adjoint of F and show that F has no right adjoint.

(3) Let R be a ring. Show that R is a division ring if and only if all R-modules are free.

(4) Let $M = \mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ and $N = \mathbb{Q}/\mathbb{Z}$, where $\mathbb{Z}[\frac{1}{p}] \subset \mathbb{Q}$ is the subring generated by $\frac{1}{p}$ for a prime p. Show that

(a) M is an artinian module but not a noetherian module;
(b) N is neither noetherian nor artinian.

(5) Let K and L be quadratic field extensions of a field k. Prove that $K \otimes_k L$ is an integral domain if and only if the k-algebras K and L are not isomorphic.

(6) Let $K \subset L$ be subfields of $\mathbb{C}$ and let p be a prime. Assume K contains a non-trivial p-th root of unity. Show that L/K is a degree p Galois extension if and only if there is an element $a \in K$ that does not admit a p-th root, such that $L = K(\sqrt[p]{a})$.

(7) Determine the ring endomorphisms of $\mathbb{F}_2[t, t^{-1}]$, where t is an indeterminate.

(8) Let G be a finite group of order pq, where p and q are distinct primes. Show that

(a) G has a normal subgroup distinct from 1 and G
(b) if $p \not\equiv 1 \pmod{q}$ and $q \not\equiv 1 \pmod{p}$, then G is abelian.

(9) Let G be a finite group of order p^n for a prime p. Show that the group ring $\mathbb{F}_p[G]$ over the finite field $\mathbb{F}_p$ with p elements has a unique maximal two-sided ideal.

(10) Let E, M and F be finite abelian groups and consider group homomorphisms $E \xrightarrow{f} M \xrightarrow{g} F$. Assume g is injective. Show that $|\mathrm{Coker}(g \circ f)| = |\mathrm{Coker}(g)| \cdot |\mathrm{Coker}(f)|$ where $|X|$ denotes the order of a finite set X.