Algebra Qualifying Exam, Fall 2016

Please do the following ten problems. Write your UID number ONLY, not your name.

(1) Let G be a group generated by a and b with only relation $a^2 = b^2 = 1$ for the group identity 1. Determine the group structure of G and justify your answer.

(2) Let K be a semi-simple quadratic extension over $\mathbb{Q}$ and consider the regular representation $\rho : K \to M_2(\mathbb{Q})$. Compute the index of $\rho(K^\times)$ in the normalizer of $\rho(K^\times)$ in $GL_2(\mathbb{Q})$, and justify your answer.

(3) Let A be an integral domain with field of fractions F. For an A-ideal $\mathfrak{a}$, prove that $\mathfrak{a}$ is an A-projective ideal finitely generated over A if there exists an A-submodule $\mathfrak{b}$ of F such that $\mathfrak{ab} = A$, where $\mathfrak{ab}$ is an A-submodule of F generated by ab for all $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$.

(4) Let D be a dihedral group of order $2p$ with normal cyclic subgroup C of order p for an odd prime p. Find the number of n-dimensional irreducible representations of D (up to isomorphisms) over $\mathbb{C}$ for each n, and justify your answer.

(5) Let $f \in F[X]$ be an irreducible separable polynomial of prime degree over a field F and let K/F be a splitting field of f. Prove that there is an element in the Galois group of K/F permuting cyclically all roots of f in K.

(6) Let F be a field of characteristic $p > 0$. Prove that for every $a \in F$, the polynomial $x^p - a$ is either irreducible or split into a product of linear factors.

(7) Let $f \in \mathbb{Q}[X]$ and $\xi \in \mathbb{C}$ a root of unity. Show that $f(\xi) \neq 2^{\frac{1}{4}}$.

(8) Prove that if a functor $\mathcal{F} : \mathcal{C} \to \mathit{Sets}$ has a left adjoint functor, then $\mathcal{F}$ is representable.

(9) Let F be a field and $a \in F$. Prove that the functor from the category of commutative F-algebras to Sets taking an algebra R to the set of invertible elements of the ring $R[X]/(X^2 - a)$ is representable.

(10) Let F be a field and A a simple subalgebra of a finite dimensional F-algebra B. Prove that $\dim_F(A)$ divides $\dim_F(B)$.