Algebra Qualifying Exam, Spring 2016

Please do the following ten problems. Write your UID number ONLY, not your name.

(1)(a) Give an example of a unique factorization domain A that is not a PID. You need not show that A is a UFD (assuming it is), but please show that your example is not a PID.

(b) Let R be a UFD. Let $\mathfrak{p}$ be a prime ideal such that $0 \neq \mathfrak{p}$ and there is no prime ideal strictly between 0 and $\mathfrak{p}$. Show that $\mathfrak{p}$ is principal.

(2) Consider the functor F from commutative rings to abelian groups that takes a commutative ring R to the group R^* of invertible elements. Does F have a left adjoint? Does F have a right adjoint? Justify your answers.

(3) Let R be a ring which is left artinian (that is, artinian with respect to left ideals). Suppose that R is a domain, meaning that $1 \neq 0$ in R and $ab = 0$ implies $a = 0$ or $b = 0$ in R. Show that R is a division ring.

(4) Let A be a commutative ring, S a multiplicatively closed subset of A, $A \to A[S^{-1}]$ the localization.

(a) Which elements of A map to zero in $A[S^{-1}]$?

(b) Let $\mathfrak{p}$ be a prime ideal in A. Show that the ideal generated by the image of $\mathfrak{p}$ in $A[S^{-1}]$ is prime if and only if the intersection of $\mathfrak{p}$ with S is empty.

(5) Let A be the ring $\mathbb{C}\langle u, v\rangle/(uv - vu - 1)$, the quotient of the free associative algebra on two generators by the given two-sided ideal.

(a) Show that every nonzero A-module M has infinite dimension as a complex vector space.

(b) Let M be an A-module with a nonzero element y such that $uy = 0$. Show that the elements $y, vy, v^2y, \ldots$ are $\mathbb{C}$-linearly independent in M.

(6) Let K be a field of characteristic $p > 0$. For an element $a \in K$, show that the polynomial $P(X) = X^p - X + a$ is irreducible over K

if and only if it has no root in K. Show also that, if P is irreducible, then any root of it generates a cyclic extension of K of degree p.

(7) Show that for every positive integer n, there exists a cyclic extension of $\mathbb{Q}$ of degree n which is contained in $\mathbb{R}$.

(8) Determine the character table of S_4, the symmetric group on 4 letters. Justify your answer.

(9) Show that if G is a finite group acting transitively on a set X with at least two elements, then there exists $g \in G$ which fixes no point of X.

(10)(a) Determine the Galois group of the polynomial $X^4 - 2$ over $\mathbb{Q}$, as a subgroup of a permutation group. Also, give generators and relations for this group.

(b) Determine the Galois group of the polynomial $X^3 - 3X - 1$ over $\mathbb{Q}$. (Hint: for polynomials of the form $X^3 + aX + b$, the quantity $\Delta = -4a^3 - 27b^2$, known as the discriminant, plays a key theoretical role.) Explain your answer.