Algebra Qualifying Exam, Fall 2017

Please do the following ten problems. Write your UID number ONLY, not your name.

(1) Let G be a finite group, p a prime number, and S a Sylow p-subgroup of G. Let $N = \{g \in G | gSg^{-1} = S\}$. Let X and Y be two subsets of $Z(S)$ (the center of S) such that there is $g \in G$ with $gXg^{-1} = Y$.

Show that there exists $n \in N$ such that $nxn^{-1} = gxg^{-1}$ for all $x \in X$.

(2) Let G be a finite group of order a power of a prime number p. Let $\Phi(G)$ be the subgroup of G generated by elements of the form g^p for $g \in G$ and $ghg^{-1}h^{-1}$ for $g, h \in G$.

Show that $\Phi(G)$ is the intersection of the maximal proper subgroups of G.

(3) Let k be a field and A a finite-dimensional k-algebra. Denote by $J(A)$ the Jacobson radical of A.

Let $t : A \to k$ be a morphism of k-vector spaces such that $t(ab) = t(ba)$ for all $a, b \in A$. Assume $\ker(t)$ contains no non-zero left ideal. Let M be the set of elements a in A such that $t(xa) = 0$ for all $x \in J(A)$.

Show that M is the largest semi-simple left A-submodule of A.

(4) Let R be a commutative noetherian ring and A a finitely generated R-algebra (not necessarily commutative). Let B be an R-subalgebra of the center $Z(A)$. Assume A is a finitely generated B-module. Show that B is a finitely generated R-algebra.

(5) Let A be a ring and M an A-module that is a finite direct sum of simple A-modules. Let $f \in \mathrm{End}_{\mathbb{Z}}(M)$. Assume $f \circ g = g \circ f$ for all $g \in \mathrm{End}_A(M)$. Consider a positive integer n.

(a). Show that the map $f_n : M^n \to M^n$ defined by $f_n(m_1, \ldots, m_n) = (f(m_1), \ldots, f(m_n))$ commutes with all elements of $\mathrm{End}_A(M^n)$.

(b). Deduce that given any family $(m_1, \ldots, m_n) \in M^n$, there exists $a \in A$ such that $(f(m_1), \ldots, f(m_n)) = (am_1, \ldots, am_n)$.

(6) Let R be an integral domain, and let M be an R-module. Prove that M is R-torsion-free if and only if the localization $M_{\mathfrak{p}}$ is $R_{\mathfrak{p}}$-torsion-free for all prime ideals $\mathfrak{p}$ of R.

(7)

(a). Show that there is at most one extension $F(\alpha)$ of a field F such that $\alpha^4 \in F$, $\alpha^2 \notin F$, and $F(\alpha) = F(\alpha^2)$.

(b). Find the isomorphism class of the Galois group of the splitting field of $x^4 - a$ for $a \in \mathbb{Q}$ with $a \notin \pm\mathbb{Q}^2$.

(8) Let F be a field, and let $f, g \in F[x] - \{0\}$ be relatively prime and not both constant. Show that $F(x)$ has finite degree $d = \max(\deg(f), \deg(g))$ over its subfield $F(\frac{f}{g})$. (Hint: If the degree were less than d, show that there exists a nonzero polynomial with coefficients in $F[x]$ of degree less than d having $\frac{f}{g}$ as a root.)

(9) Let R be a commutative ring, and let A, B, and C be R-modules. Suppose that A is finitely presented over R and C is flat over R. Show that

$$\mathrm{Hom}_R(A, B \otimes_R C) \cong \mathrm{Hom}_R(A, B) \otimes_R C.$$

(10) Let $\mathcal{C}$ be a category with finite products, and let $\mathcal{C}^2$ be the category of pairs of objects of C together with morphisms $(A, A') \to (B, B')$ of pairs consisting of pairs $(A \to B, A' \to B')$ of morphisms in $\mathcal{C}$. Let $F\colon \mathcal{C}^2 \to \mathcal{C}$ be the direct product functor (that takes pairs of objects and morphisms to their products).

a. Find a left adjoint to F.

b. For $\mathcal{C}$ the category of abelian groups, determine whether or not F has a right adjoint.