Algebra Qualifying Exam, Spring 2017

Please do the following ten problems. Write your UID number ONLY, not your name.

(1) Choose a representative for every conjugacy class in the group $GL(2, \mathbb{R})$. Justify your answer.

(2) Let G be the group with presentation

$$\langle x, y : x^4 = 1, y^5 = 1, xyx^{-1} = y^2 \rangle,$$

which has order 20. Find the character table of G.

(3) Find the number of subgroups of index 3 in the free group $F_2 = \langle u, v \rangle$ on two generators. Justify your answer.

(4) Show that the ring $R = \mathbb{C}[x, y]/(y^2 - x^3 + 1)$ is a Dedekind domain. (Hint: compare R with the subring $\mathbb{C}[x]$.)

(5) Let S be a multiplicatively closed subset of a commutative ring R. For a prime ideal I in R with $I \cap S = \emptyset$, show that the ideal $I \cdot R[S^{-1}]$ in the localized ring $R[S^{-1}]$ is prime. Also, show that sending I to $I \cdot R[S^{-1}]$ gives a bijection between the prime ideals in R that do not meet S and the prime ideals in the localized ring $R[S^{-1}]$.

(6) Prove the following generalization of Nakayama's Lemma to noncommutative rings. Let R be a ring with 1 (not necessarily commutative) and suppose that $J \subset R$ is a left ideal contained in every maximal left ideal of R. If M is a finitely generated left R-module such that $JM = M$, prove that $M = 0$.

(7) Find $[K : \mathbb{Q}]$ where K is a splitting field of $X^6 - 4X^3 + 1$ over $\mathbb{Q}$.

(8) Let M be an abelian group (written additively). Prove that there is a functor F from the opposite of the category of rings to the category of sets taking a ring R to the set of all left R-module structures on M. Is the functor F representable?

(9) Let R be a ring. Prove that if the left free R-modules R^n and R^m are isomorphic for some positive integers n and m, then R^n and R^m are isomorphic as right R-modules.

(10) Let K/F be a (finite) Galois field extension with $G = \mathrm{Gal}(K/F)$ and let $H \subset G$ be a subgroup. Determine in terms of H and G the group $\mathrm{Gal}(K^H/F)$ of all field automorphisms of K^H over F.