ALGEBRA QUALIFYING EXAM

2018 SEPTEMBER 11

Instructions: Solve 10 of the 12 problems. If you solve more than 10, indicate clearly which 10 you want graded. State clearly the theorems you use.

Write your University ID number (*not* your name) on every page you turn in.

* * *

Problem 1. Let $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group of order 8.

(1) Show that every non-trivial subgroup of Q_8 contains -1.
(2) Show that Q_8 does not embed in the symmetric group S_7 (as a subgroup).

Problem 2. Let G be a finitely generated group having a subgroup of finite index $n > 1$. Show that G has finitely many subgroups of index n and has a proper characteristic subgroup (*i.e.* preserved by all automorphisms) of finite index.

Problem 3. Let K/F be a finite extension of fields. Suppose that there exist finitely many intermediate fields $K/E/F$. Show that $K = F(x)$ for some $x \in K$.

Problem 4. Let K be a subfield of the real numbers and f an irreducible degree 4 polynomial over K. Suppose that f has exactly two real roots. Show that the Galois group of f is either S_4 or of order 8.

Problem 5. Let R be a commutative ring. Show the following:

(a) Let S be a non-empty *saturated* multiplicative set in R, *i.e.* if $a, b \in R$, then $ab \in S$ if and only if $a, b \in S$. Show that $R \setminus S$ is a union of prime ideals.
(b) If R is a domain, show that R is a UFD if and only if every nonzero prime ideal in R contains a non-zero principal prime ideal.

Problem 6. Let A be an integrally closed Noetherian domain with quotient field F and K/F be a finite separable field extension.

(a) If $\{x_1, \ldots, x_n\}$ is a basis for K as an F-vector space, show that there exists $\{y_1, \ldots, y_n\}$ in K such that $\mathrm{Tr}_{K/F}(x_i y_j) = \delta_{i,j}$ for all i, j.
(b) If B is the integral closure of A in K, show that B is a finitely generated A-module.

Problem 7. Let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor with a right adjoint G. Show that F is fully faithful if and only if the unit of the adjunction $\eta \colon \mathrm{Id}_{\mathcal{C}} \to GF$ is an isomorphism.

Problem 8. Give an example of a diagram of commutative rings whose colimit in the category of commutative rings is different from its colimit in the larger category of rings (and ring homomorphisms).

Problem 9. Let $f \colon M \to N$ and $g \colon N \to M$ be two R-linear homomorphisms of R-modules such that $\mathrm{id}_M - gf$ is invertible. Show that $\mathrm{id}_N - fg$ is invertible as well and give a formula for its inverse. [Hint: You may use Analysis to make a guess.]

Problem 10. Consider the real algebra $A = \mathbb{R}[x, y] = \mathbb{R}[X, Y]/X^2 + Y^2 - 1$ where x and y are the classes of X and Y respectively. Let $M = A(1 + x) + Ay$ be the ideal generated by $1 + x$ and y. (This is the *Möbius band.*)

(1) Show that there is an A-linear isomorphism $A^2 \overset{\sim}{\to} M \oplus M$ mapping the canonical basis to $(1 + x, y)$ and $(-y, 1 + x)$.
(2) Show that there is an A-linear isomorphism $A \overset{\sim}{\to} M \otimes_A M$ mapping 1 to $((1 + x) \otimes (1 + x)) \ + \ (y \otimes y)$.

Problem 11. Let G be a finite group, ω be a primitive 3rd root of 1 in $\mathbb{C}$ and suppose that the complex character table of G contains the row

$$1 \quad \omega \quad \omega^2 \quad 1.$$

Determine the whole complex character table of G, the order of the group and the order of its conjugacy classes.

Problem 12. Let F be a finite field and $K \subset \bar{F}$ the subfield of an algebraic closure generated by all roots of unity. Find all simple finite dimensional K-algebras.