Algebra Qualifying Exam, Spring 2018

Please do the following ten problems. Write your UID number ONLY, not your name.

(1) Let $\alpha \in \mathbb{C}$. Suppose that $[\mathbb{Q}(\alpha) : \mathbb{Q}]$ is finite and prime to $n!$ for an integer $n > 1$. Show that $\mathbb{Q}(\alpha^n) = \mathbb{Q}(\alpha)$.

(2) Let $\zeta^9 = 1$ and $\zeta^3 \neq 1$ with $\zeta \in \mathbb{C}$.
(a) Show that $\sqrt[3]{3} \notin \mathbb{Q}(\zeta)$,
(b) If $\alpha^3 = 3$, show that α is not a cube in $\mathbb{Q}(\zeta, \alpha)$.

(3) Let $\mathbb{Z}^n$ $(n > 1)$ be made of column vectors with integer coefficients. Prove that for every non-zero left ideal I of $M_n(\mathbb{Z})$, $I\mathbb{Z}^n$ (the subgroup generated by products αv with $\alpha \in I$ and $v \in \mathbb{Z}^n$) has finite index in $\mathbb{Z}^n$.

(4) Let p be a prime number, and let D be a central simple division algebra of dimension p^2 over a field k. Pick $\alpha \in D$ not in the center and write K for the subfield of D generated by α. Prove that $D \otimes_k K \cong M_p(K)$ (the $p \times p$ matrix algebra with entries in K).

(5) Let C be a category. A morphism $f : A \to B$ in C is called an *epimorphism* if for any two morphisms $g, h : B \to X$ in C, $g \circ f = h \circ f$ implies $g = h$. Let ALG be the category of $\mathbb{Z}$-algebras, and let MOD be the category of $\mathbb{Z}$-modules.
(a) Prove that in MOD, $f : M \to N$ is an epimorphism if and only if f is a surjection.
(b) In ALG, does the equivalence of epimorphism and surjection hold? If yes, prove the equivalence, and if no, give a counterexample (as simple as possible).

(6) Let G be a group with a normal subgroup $N = \langle y, z \rangle$ isomorphic to $(\mathbb{Z}/2)^2$. Suppose that G has a subgroup $Q = \langle x \rangle$ isomorphic to the cyclic group $\mathbb{Z}/3$ such that the composition $Q \subset G \to G/N$ is an isomorphism. Finally, suppose that $xyx^{-1} = z$ and $xzx^{-1} = yz$. Compute the character table of G.

(7) Let B be a commutative noetherian ring, and let A be a noetherian subring of B. Let I be the nilradical of B. If B/I is finitely generated as an A-module, show that B is finitely generated as an A-module.

(8) Let F be a field that contains the real numbers $\mathbb{R}$ as a subfield. Show that the tensor product $F \otimes_{\mathbb{R}} \mathbb{C}$ is either a field or isomorphic to the product of two copies of F, $F \times F$.

(9) Show that there is no simple group of order 616.

(10) By one definition, a *Dedekind domain* is a commutative noetherian integral domain R, integrally closed in its fraction field, such that R is not a field and every nonzero prime ideal in R is maximal. Let R be a Dedekind domain, and let S be a multiplicatively closed subset of R. Show that the localization $R[S^{-1}]$ is either the zero ring, a field, or a Dedekind domain.