Algebra Qualifying Exam

Fall 2019

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1. Show that every group of order 315 is the direct product of a group of order 5 with a semidirect product of a normal subgroup of order 7 and a subgroup of order 9. How many such isomorphism classes are there?

2. Let L be a finite Galois extension of a field K inside an algebraic closure $\bar{K}$ of K. Let M be a finite extension of K in $\bar{K}$. Show that the following are equivalent:

   i. $L \cap M = K$,

   ii. $[LM : K] = [L : K][M : K]$,

   iii. every K-linearly independent subset of L is M-linearly independent.

3. Let I be the ideal $(x^2 - y^2 + z^2, (xy + 1)^2 - z, z^3)$ of $R = \mathbb{C}[x, y, z]$. Find the maximal ideals of R/I, as well as all of the points on the variety

$$V(I) = \{(a, b, c) \in \mathbb{C}^3 \mid f(a, b, c) = 0 \text{ for all } f \in I\}.$$

4. Find all isomorphism classes of simple (i.e., irreducible) left modules over the ring $M_n(\mathbb{Z})$ of n-by-n matrices with $\mathbb{Z}$-entries with $n \geq 1$.

5. Let R be a nonzero commutative ring. Consider the functor t_B from the category of R-modules to itself given by taking the (right) tensor product with an R-module B.

   a. Prove that t_B commutes with colimits.

   b. Construct an R-module B (for each R) such that t_B does not commute with limits in the category of R-modules.

6. Classify all finite subgroups of $\mathrm{GL}(2, \mathbb{R})$ up to conjugacy.

7. Let G be the group of order 12 with presentation

$$G = \langle g, h \mid g^4 = 1, h^3 = 1, ghg^{-1} = h^2 \rangle.$$

Find the conjugacy classes of G and the values of the characters of the irreducible complex representations of G of dimension greater than 1 on representatives of these classes.

8. Let M be a finitely generated module over an integral domain R. Show that there is a nonzero element $u \in R$ such that the localization $M[1/u]$ is a free module over $R[1/u]$.

9. Let A be a unique factorization domain which is a $\mathbb{Q}$-algebra. Let K be the fraction field of A. Let L be a quadratic extension field of K. Show that the integral closure of A in L is a finitely generated free A-module.

10. Compute the Galois groups of the Galois closures of the following field extensions:

a. $\mathbb{C}(x)/\mathbb{C}(x^4+1)$,

b. $\mathbb{C}(x)/\mathbb{C}(x^4+x^2+1)$,

where $\mathbb{C}(y)$ denotes the field of rational functions over $\mathbb{C}$ in a variable y.