ALGEBRA QUALIFYING EXAM

Instructions: Do the following ten problems. Write your UID number only, not your name.

1. Let G be a finite solvable group and $1 \neq N \subset G$ be a minimal normal subgroup. Prove that there exists a prime p such that N is either cyclic of order p or a direct product of cyclic groups of order p.

2. An additive group (abelian group written additively) Q is called *divisible* if any equation $nx = y$ with $0 \neq n \in \mathbb{Z}$, $y \in Q$ has a solution $x \in Q$. Let Q be a divisible group and A is a subgroup of an abelian group B. Give a complete proof of the following: every group homomorphism $A \to Q$ can be extended to a group homomorphism $B \to Q$.

3. Let $d > 2$ be a square-free integer. Show that the integer 2 in $\mathbb{Z}[\sqrt{-d}]$ is irreducible but the ideal (2) in $\mathbb{Z}[\sqrt{-d}]$ is not a prime ideal.

4. Let R be a commutative local ring and P a finitely generated projective R-module. Prove that P is R-free.

5. Let Φ_n denote the nth cyclotomic polynomial in $\mathbb{Z}[X]$ and let a be a positive integer and p a (positive) prime not dividing n. Prove that if $p \mid \Phi_n(a)$ in $\mathbb{Z}$, then $p \equiv 1 \mod n$.

6. Let F be a field of characteristic $p > 0$ and $a \in F^{\times}$. Prove that if the polynomial $f = X^p - a$ has no root in F, then f is irreducible over F.

7. Let F be a field and let R be the ring of 3×3 matrices over F with $(3, 1)$ and $(3, 2)$ entry equal to 0. Thus,

$$R = \begin{bmatrix} F & F & F \\ F & F & F \\ 0 & 0 & F \end{bmatrix}.$$

(a) Determine the Jacobson radical J of R.

(b) Is J a minimal left (respectively, right) ideal?

8. Prove that every finite group of order n is isomorphic to a subgroup of $GL_{n-1}(\mathbb{C})$.

9. a) Find a domain R and two nonzero elements $a, b \in R$ such that R is equal to the intersection of the localizations $R[1/a]$ and $R[1/b]$ (in the quotient field of R) and $aR + bR \neq R$.
b) Let $\mathcal{C}$ be the category of commutative rings. Prove that the functor $\mathcal{C} \to Sets$ taking a commutative ring R to the set of all pairs $(a, b) \in R^2$ such that $aR + bR = R$ is not representable.

10. Let $\mathcal{C}$ be an abelian category. Prove that TFAE:
(1) Every object of $\mathcal{C}$ is projective.
(2) Every object of $\mathcal{C}$ is injective.