Algebra Qualifying Exam Fall 2020

All rings have a one but are not necessarily commutative.

1. Let $p < q < r$ be primes and G a group of order pqr. Prove that G is not simple and, in fact, has a normal Sylow r-group.

2. Show that groups of order 231 =(3)(7)(11) are semi-direct products and show that there are exactly two such groups up to isomorphism.

3. A ring R (commutative or non-commutative) is called a *domain* if $ab = 0$ in R implies $a = 0$ or $b = 0$. Suppose that R is a domain such that $\mathbb{M}_n(R)$, the ring of $n \times n$ matrices over R, is a semisimple ring. Prove that R is a division ring.

4. Let M be a left R-module. Show that M is a projective R-module if and only if there exist $m_i \in M$ and R-homomorphisms $f_i : M \to R$ for each $i \in I$ such that the sets $\{m_i \mid i \in I\}$ and $\{f_i \mid i \in I\}$ satisfy:

   (a) If $m \in M$, then $f_i(m) = 0$ for all but finitely many $i \in I$.

   (b) If $m \in M$, then $m = \sum_{i \in I} f_i(m)m_i$.

5. Let F be a field and $f(X) = X^6 + 3 \in F[X]$. Determine a splitting field K of $f(X)$ over F and determine $[K : F]$ and $\mathrm{Gal}(K/F)$ for each of the following three fields: $F = \mathbb{Q}$, $\mathbb{F}_5$, $\mathbb{F}_7$.

6. Let $K_1 \subset K_2 \subset K_3$ be fields with K_3/K_2 and K_2/K_1 both Galois. Let L be a minimal Galois extension of K_1 containing K_3. Show if the Galois groups $\mathrm{Gal}(K_3/K_2)$ and $\mathrm{Gal}(K_2/K_1)$ are both p-groups so is the Galois group $\mathrm{Gal}(L/K_1)$.

7. Let R be a Dedekind domain with quotient field K and I a nonzero ideal in R. Show both of the following:

   (a) Every ideal in R/I is a principal ideal.

   (b) If J is a fractional ideal of R, i.e., $0 \neq J \subset K$ is an R-module such that there exists a $d \in R$ with $dJ \subset R$, then there exists a $0 \neq x$ in K such that $I + xJ = R$.

8. Consider $R = \mathbb{C}[X, Y]/(X^2, XY)$. Determine the prime ideals P of R. Which of the localizations R_P are integral domains?

9. Let G be a finite group, F a field, and V a finite dimensional F-vector space with $G \to \mathrm{GL}(V)$ a faithful irreducible representation. Show that the center $Z(G)$ of G is cyclic.

10. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and suppose that every pair of morphisms in $\mathcal{C}$ admits a coequalizer. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor that preserves coequalizers: i.e., if $f, g\colon A \to B$ are morphisms in $\mathcal{C}$ and $\pi\colon B \to \mathrm{coeq}(f, g)$ is the coequalizer morphism, then $F(\pi)$ is a coequalizer morphism for $F(f)$ and $F(g)$. Suppose also that if h is a morphism in $\mathcal{C}$ such that $F(h)$ is an isomorphism, then h is an isomorphism. Show that F is faithful.