ALGEBRA QUALIFYING EXAM

2020 JUNE

All answers must be justified. State clearly theorems that you use.

Problem 1. Let G be a group defined by $G = \langle a, b \mid a^2 = b^2 = 1 \rangle$. Determine the order of all non-trivial finite quotient groups.

Problem 2. Let G be a finite group of order $n > 1$ and consider its group algebra $\mathbb{Z}[G]$ embedded in $\mathbb{Q}[G]$. Let $A = \mathbb{Z}[G]/\mathfrak{a}$ for the ideal $\mathfrak{a}$ generated by $g - 1$ for all $g \in G$.

(a) Prove that the algebra $\mathbb{Q}[G]$ is the product of $\mathbb{Q}$ and $\mathbb{Q} \cdot \mathfrak{a}$, where $\mathbb{Q} \cdot \mathfrak{a}$ is the $\mathbb{Q}$-span of $\mathfrak{a}$ in $\mathbb{Q}[G]$. [Hint: First identify the unit $1_{\mathbb{Q}\cdot\mathfrak{a}}$.]

(b) Let B be the projected image of $\mathbb{Z}[G]$ in $\mathbb{Q} \cdot \mathfrak{a}$. Prove that

$$A \otimes_{\mathbb{Z}[G]} B \cong G$$

as groups if and only if G is a cyclic group.

Problem 3. Prove that a noetherian commutative ring A is a finite ring if the following two conditions are satisfied:

(i) the nilradical of A vanishes,

(ii) localization at every maximal ideal is a finite ring.

Problem 4. Compute the dimension of the tensor products of two algebras $\mathbb{Q}[\sqrt{2}] \otimes_{\mathbb{Z}} \mathbb{Q}[\sqrt{2}]$ over $\mathbb{Q}$ and $\mathbb{Q}[\sqrt{2}] \otimes_{\mathbb{Z}} \mathbb{R}$ over $\mathbb{R}$. Is $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}$ finite dimensional over $\mathbb{R}$?

Problem 5. If $K \neq \mathbb{Q}$ appears as a subfield (sharing the identity) of some central simple algebra over $\mathbb{Q}$ of $\mathbb{Q}$-dimension 9, determine (isomorphism classes of) the groups appearing as the Galois group of the Galois closure of K over $\mathbb{Q}$.

Problem 6. Let $\mathbb{F}$ be a finite field with at least 3 elements. Show that $\mathrm{SL}_2(\mathbb{F})$ has order divisible by 12.

Problem 7. Let G be a p-group and $1 \neq N \trianglelefteq G$ be a non-trivial normal subgroup.

(a) Show that N contains a non-trivial element of the center $Z(G)$ of G.

(b) Give an example where $Z(N) \not\subseteq Z(G)$.

Problem 8. Let R be a ring.

(a) Show that an R-module X is indecomposable if $\mathrm{End}_R(X)$ is *local*. (Recall that a ring is local if the sum of non-invertible elements remains non-invertible).

(b) Suppose that every finitely generated R-module M is isomorphic to $X_1 \oplus \cdots \oplus X_m$ with all $\mathrm{End}_R(X_i)$ local. Show that such a decomposition is unique: If $X_1 \oplus \cdots \oplus X_m \simeq Y_1 \oplus \cdots \oplus Y_n$ then $m = n$ and there is a bijection $\sigma \in S_n$ and isomorphisms $X_i \simeq Y_{\sigma(i)}$.

(c) Give an example of an isomorphism $X_1 \oplus X_2 \simeq Y_1 \oplus Y_2$ with $\mathrm{End}(X_i)$ and $\mathrm{End}(Y_i)$ local that is *not* the direct sum of any isomorphisms $X_i \simeq Y_i$, even up to renumbering the Y_i.

Problem 9. Let R be a commutative ring and $S \subset R$ a multiplicative subset. Construct a natural transformation (in either direction) between the functors $\mathrm{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N)$ and $S^{-1}\mathrm{Hom}_R(M, N)$, considered as functors of R-modules M and N, and prove it is an isomorphism if M is finitely presented.

Problem 10. Let R be a commutative ring and M a left R-module. Let $f\colon M \to M$ be a surjective R-linear endomorphism. [Hint: Let $R[X]$ acts on M via f.]

(a) Suppose that M is finitely generated. Show that f is an isomorphism and that f^{-1} can be described as a polynomial in f.

(b) Show that this fails if M is not finitely generated.