ANALYSIS QUALIFYING EXAM, FALL 2001

Directions:

All problems on this test are worth 10 points. There are six Real Analysis problems and five Complex Analysis problems.

Your score will be computed from your best scores on **five** Real Analysis problems and **five** Complex Analysis problems.

In this exam you may use the axiom of choice.

Real Analysis Problems

R1: Consider real numbers $a_{n,m}$ for $n = 1, 2, \ldots$ and $m = 1, 2, \ldots$ and assume that inner and outer sums in the expressions

$$A := \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} a_{n,m} \right]$$

$$B := \sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} a_{n,m} \right]$$

are absolutely convergent.

a) Give an example that shows that we may have $A \neq B$.

b) Under what reasonable additional assumption on $a_{n,m}$ can we conclude $A = B$?

R2: Let $n \geq 1$. Let $O(n)$ denote the set of all real $n \times n$ matrices G which satisfy

$$G^T G = I$$

where I is the identity matrix (O stands for "orthogonal group").

a) Prove that $O(n)$ is compact.

b) Prove that $O(n)$ is not connected.

R3: Prove the mean value theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. For every $a < b$ there exists $a < \xi < b$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(\xi)$$

R4: The Fourier transform $\widehat{f}$ of a function f in $L^1(\mathbb{R})$ is defined as

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x \xi} f(x)\, dx$$

a) State how the Fourier transform of a function in $L^2(\mathbb{R})$ is defined. (You do not need to prove claims which you use to state this. Do not use the fact b) below.)

b) Let f be in $L^2(\mathbb{R})$. Define

$$Mf(\xi) = \sup_{r>0} \left| \int_{-r}^{r} f(x) e^{-2\pi i x \xi}\, dx \right|$$

A deep fact proved by Carleson and Hunt states that

$$||Mf||_2 \leq C||f||_2$$

for some universal constant C. Use this theorem to prove

$$\widehat{f}(\xi) = \lim_{r\to\infty} \int_{-r}^{r} e^{-2\pi i x \xi} f(x)\, dx$$

for almost every $\xi \in \mathbb{R}$.

R5: For $n \geq 1$ let $l^1(n)$, $l^2(n)$, $l^\infty(n)$ be the Banach space $\mathbb{R}^n$ equipped with the norm

$$\sum_{i=1}^{n} |x_i|, \quad (\sum_{i=1}^{n} |x_i|^2)^{1/2}, \quad \sup_{i=1,\dots,n} |x_i|$$

respectively. Answer the following questions and prove your assertions:

a) Which of the Banach spaces $l^1(2)$, $l^2(2)$, $l^\infty(2)$ are isometrically isomorphic?

b) Which of the Banach spaces $l^1(3)$, $l^2(3)$, $l^\infty(3)$ are isometrically isomorphic?

R6: Let V be the complex Banach space $l^\infty(\mathbb{N})$, i.e. the space of all sequences $x = (x_n)_{n=1,2,\dots}$ with the norm $||x|| = \sup_n |x_n|$. Every sequence f in $l^1(\mathbb{N})$ gives rise to a linear functional $\phi_f : V \to \mathbb{C}$ by the formula $\phi_f(x) = \sum_{n=1}^{\infty} f_n x_n$

a) Prove that ϕ_f is continuous for each $f \in l^1(\mathbb{N})$.

b) Prove that there are elements in the dual space of V which are not of the form ϕ_f for any $f \in l^1(\mathbb{N})$. Hint: consider the subspace of V consisting of convergent sequences.

Complex Analysis Problems

C1: Find an explicit conformal mapping from the region

$$\{|z| < 1\} \setminus [0, 1)$$

onto the upper half plane $\{Imz > 0\}$.

C2:
Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos x\, dx}{9 + 10x^2 + x^4} .$$

C3:
Define

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

and

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

a) For m and n positive integers, calculate

$$\frac{\partial}{\partial \overline{z}} z^n \overline{z}^m \quad .$$

b) Let $P(x,y)$ be a polynomial in the two real variables x and y. Then P has the form

$$P(x,y) = \sum_{n=0}^{N}\sum_{m=0}^{N} a_{n,m} z^n \overline{z}^m$$

but you do not need to prove it. Prove however that if

$$\frac{\partial}{\partial \overline{z}} P(x,y) = 0 \quad ,$$

then $a_{n,m} = 0$ or all $m > 0$.

C4:
Let $u(z)$ be a harmonic function on the entire plane $\mathbb{C}$ such that

$$\int\int_{\mathbb{C}} |u(z)|^2 \, dxdy < \infty \quad .$$

Prove $u(z) = 0$ for all z.

C5:
Let $F(z)$ be continuous on the closed unit disc $\overline{\mathbb{D}} = \{z : |z| \leq 1\}$ and analytic on the open disc $\mathbb{D} = \{z : |z| < 1\}$.

a) Prove

$$\lim_{\lambda \uparrow 1} F(\lambda z) = F(z)$$

uniformly on $\overline{\mathbb{D}}$

b) If also

$$F(z) = \sum_{n=0}^{\infty} a_n z^n$$

on $\mathbb{D}$, prove that

$$\lim_{n\to\infty} a_n = 0 \quad .$$