New/revised

ANALYSIS QUALIFYING EXAMINATION

SEPTEMBER 18, WEDNESDAY, 2002

2:00 - 6:00 PM

ROOM: MS 5200

Instructions

Work any 10 problems, but must include 2 problems each from Part I, Part II and Part III respectively. All problems are worth 10 points.

Part I

1. Let f, g be two absolutely continuous functions on the interval $[0,1]$ which are everywhere positive. Show that the pointwise quotient f/g is also absolutely continuous on $[0,1]$. [**Hint**: Is g bounded from below?]

2. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let f be a real-valued function in $L^1(X, \mathcal{M}, \mu)$. Show that there exists a non-negative function $f^* \in L^1([0,\infty))$ which is monotone non-increasing (i.e. $f^*(x) \leq f^*(y)$ for all $x \geq y \geq 0$), and such that

$$\int_X |f|^p \mathrm{d}\mu = \int_0^\infty f^*(x)^p \, \mathrm{d}x$$

for all $0 < p < \infty$. [**Hint**: Choose f^* so that $\mu(\{x \in X : |f(x)| \geq \lambda\}) = m(\{y \in [0,\infty) : f^*(y) \geq \lambda\})$ for all $\lambda > 0$, where m is Lebesgue measure.]

3. Let $(X, \mathcal{M}, \mu)$ be a finite measure space, and let $1 \leq p < \infty$. Let $f_1, f_2, \ldots$ be a sequence of functions in $L^p(X, \mathcal{M}, \mu)$ which converge pointwise μ-a.e. to a function $f \in L^p(X, \mathcal{M}, \mu)$. Show that $\lim_{n\to\infty} \|f_n - f\|_p = 0$ if and only if $\lim_{n\to\infty} \|f_n\|_p = \|f\|_p$.

4. Let E and F be two Lebesgue measurable sets of the real line of finite measure, and let χ_E and χ_F be their respective characteristic functions.

(a) Show that the convolution $\chi_E * \chi_F$, defined by

$$\chi_E * \chi_F(x) = \int_{\mathbf{R}} \chi_E(y)\chi_F(x-y) \, dy$$

is a continuous function.

(b) Show that $\chi_E * \chi_F$ lies in L^p for every $1 \leq p \leq \infty$.

5. Let $0 < \alpha < 1$. A function $f \in C([0,1])$ is said to be *Hölder continuous of order* α if there exists a constant C such that

$$|f(x) - f(y)| \leq C|x-y|^\alpha$$

for all $x, y \in [0,1]$.

Show that for every $0 < \alpha < 1$, the function

$$f(x) = \sum_{n=1}^{\infty} 2^{-n\alpha} \cos(2^n x)$$

is Hölder continuous of order α but is nowhere differentiable on $[0,1]$.

6. Let $n > 1$ be an integer, and let $B = \{x \in \mathbf{R}^n : |x| < 1\}$ be the open unit ball in $\mathbf{R}^n$. Show that there exists a constant $0 < C < \infty$ depending only on n, such that

$$\int_B |u(x)|^2 \, dx \leq C \int_B |\nabla u(x)|^2 dx$$

for all smooth, compactly supported real-valued functions $u : B \to \mathbf{R}$.

Part II

7. Show that for every Hilbert space H and every closed convex subset $\Omega \subset H$, there exists a unique element $x \in \Omega$ of minimal norm, i.e., $\|x\| \leq \|y\|$ for all $y \in \Omega$.

8. Show that the set

$$\Omega = \left\{ f \in C([-1,1]) : \int_{-1}^{1} f(x) signum(x) \, dx = 1 \right\}$$

is a close convex set Ω in the Banach space $C([-1,1])$ which does not contain any element of minimal norm. Here $signum(x)$ is the function which equals $+1$ for positive x, -1 for negative x, and 0 for $x = 0$.

[**Hint**: If $C([-1,1])$ were replaced by the larger space $L^\infty([-1,1])$, what would be the element of minimal norm?]

9. Let $T : L^2([0,1]) \to L^2([0,1])$ be the operator

$$Tf(x) = \int_0^x f(y) \, dy.$$

(a) Show that the Fourier coefficients

$$\widehat{Tf}(n) = \int_0^1 e^{-2\pi i n x} Tf(x) \, dx$$

of Tf obey the bound

$$|\widehat{Tf}(n)| \leq \frac{C\|f\|_{L^2([0,1])}}{n}$$

for all non-zero n, all $f \in L^2([0,1])$, and some constant $0 < C < \infty$.

(b) Show that T is a continuous, compact operator (i.e. the image of the closed unit ball is compact). [**Hint**: use (a)].

(c) Show that for any complex non-zero λ, the operator $T - \lambda$ has no kernel (i.e. there is no non-zero $f \in L^2([0,1])$ such that $(T-\lambda)f = 0$).

(d) Show that the operator $T - \lambda$ is invertible on $L^2([0,1])$ for all complex non-zero λ. [**Hint**: Use (b) and (c)].

Part III

10. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $f_1, f_2, \ldots$ be sequence of complex-valued functions in $L^1(X, \mathcal{M}, \mu)$ such that

$$\|f_n\|_{L^1(X,\mathcal{M},\mu)} \leq 2^{-n}$$

and

$$\|f_n\|_{L^\infty(X,\mathcal{M},\mu)} \leq 1/2$$

for all $n = 1, 2, \ldots$. Show that the infinite product

$$\prod_{n=1}^{\infty} (1 + f_n(x))$$

is convergent for μ-a.e. x, and that the product is a measurable function. [**Hint**: You may need to compare $|1+z|$ and $\exp(|z|)$ in the disk $|z| \leq 1/2$.]

11. Find the linear fractional tranformation $f : \mathbf{C} \to \mathbf{C}$ such that $f(0) = 1$, $f(1) = 0$, and $f(\infty) = i$. What is the image of the line $\{z : \mathrm{Re}(z) = 1\}$ under f?

12. Suppose the function $f(z)$ is continuous on the closed unit disk $\{|z| \leq 1\}$ and analytic on the open disk $\{|z| < 1\}$. Assume $f(z) = 0$ for all z in the semi-circle $\{z : |z| = 1, \mathrm{Im}(z) > 0\}$. Prove that $f(z) = 0$ on the closed disk.

13. Let $U_n(z)$ be a sequence of *positive* harmonic functions on a *connected* open set Ω containing the origin. Show that if

$$\lim_{n \to \infty} U_n(0) = 0,$$

then

$$\lim_{n \to \infty} \|U_n\|_{L^\infty(K)} = 0$$

for all compact subsets $K \subset \Omega$.

14. Evaluate the integral

$$\lim_{N\to\infty}\int_0^N \cos(x^2)\mathrm{d}x;$$

justify your reasoning.

15. Let $1 \le p < \infty$ and let $U(z)$ be a harmonic function on the entire plane $\mathbf{C}$ such that

$$\int\int_{\mathbf{C}} |U(z)|^p \mathrm{d}x\mathrm{d}y < \infty.$$

Prove $U(z) = 0$ for all z.