Analysis Qualifying Examination - May 18, 2002

Instructions:

Work any 12 problems, and especially three from Problems 9 to 14. All problems are worth ten points.

1. Let V be a finite dimensional real vector space, and let $|| \; ||_V$ be a norm on V. Let $\mathcal{P}$ be the set of one-dimensional linear subspaces of V ($\mathcal{P}$ is called a real projective space.) For $W_1, W_2 \in \mathcal{P}$ define

$$d(W_1, W_2) = \inf\{||v_1 - v_2||_V : v_j \in W_j \text{ and } ||v_j||_V = 1\}.$$

Prove that d is a metric on $\mathcal{P}$ and that $\mathcal{P}$ is compact with respect to this metric.

2. Let $\{a_n\}_n$ be a sequence of real numbers such that $\lim_n a_n = 0$ but such that $\sum_n a_n$ is divergent. Show that for any real number r there is a sequence $\sigma_n \in \{-1, 1\}$ such that

$$\sum_n \sigma_n a_n = r.$$

3. Prove or disprove that there exists an infinite dimensional real Banach space X containing a countable subset S such that every $x \in X$ is a (finite) linear combination of elements of S.

4. Let $f_n(x)$ be a sequence of Borel measurable real-valued functions on the interval $[0,1]$ such that $f_n(x) \to 0$ for all $x \in [0,1]$. Let $\epsilon > 0$. Prove there is a Borel set $A \subset [0,1]$ such that

(i) $m([0,1] \cap A') < \epsilon$;

where m is Lebesgue measure, and

(ii) $f_n(x) \to 0$ uniformly on A.

5. Let $(X, \mathcal{M}, \mu)$ be a measure space such that $\mu(X) = 1$. Let $f \in L^1(X, \mathcal{M}, \mu)$. Prove

$$\lim_{p \to 0} \Big(\int |f|^p d\mu \Big)^{1/p} = \exp\Big(\int \log |f| d\mu \Big)$$

where $e^{-\infty}$ is defined to be 0. Hint: Jensen's inequality.

6. On the two-point space $\{0, 1\}$ let μ be the measure such that $\mu(\{0\}) = \mu(\{1\}) = 1/2$. Let X be the infinite product space

$$\prod_{j=1}^{\infty} X_j$$

where each $X_j = \{0, 1\}$, and let σ be the product measure on X (defined by $\sigma\left(\bigcap_{j=1}^{n}\{x : x_j = a_j\}\right) = 2^{-n}$). Find an explicit mapping $f : X \to [0, 1]$ such that there exists $X' \subset X$, with $\mu(X') = 0$ and $Y' \subset [0, 1]$ with Lebesgue measure zero such that

$$f : X \setminus X' \to [0, 1] \setminus Y'$$

is one-to-one and measure preserving, i. e. $m(F(E)) = \mu(E)$ where m denotes Lebesque measure.

7. a). For x and ξ real, show that every partial sum of the series

$$e^{ix\xi} = \sum_{n=0}^{\infty} \frac{i^n x^n \xi^n}{n!}$$

is bounded by $e^{|x||\xi|}$.

(b). Let $\mathcal{H}$ be the Hilbert space of Lebesque measureable functions on the real line with inner product

$$< f, g >= \int_{-\infty}^{\infty} f(x)\overline{g(x)}e^{-x^2}\,dx.$$

Prove that the set of polynomials $a_0 + a_1x + ... + a_nx^n$ is dense in $\mathcal{H}$. Hint: Suppose $g \in \mathcal{H}$ is orthogonal to the polynomials. Show that $G(x) = g(x)e^{-x^2} \in L^1(\mathbb{R}, dx)$ and that

$$\hat{G}(\xi) = \int e^{ix\xi}G(x)dx = 0$$

for all $\xi \in \mathbb{R}$. Use this information to show $g = 0$ almost everywhere.

8. Let H be the Hilbert space $L^2(\mathbb{R})$ of square (Lebesgue) integrable functions on the line $\mathbb{R}$ and define $U : H \to H$ by

$$Uf(x) = f(x - 1).$$

Show that U has no (non-zero) eigenvectors.

9. Let f be any conformal mapping from the strip $S = \{z \in \mathbb{C} : -1 < \Im z < 1\}$ onto the unit disc $\mathbb{D} = \{z : |z| < 1\}$ such that uniformly in $y \in (-1, 1)$,

$$\lim_{x\to\infty} f(x + iy) = 1, \text{ and } \lim_{x\to-\infty} f(x + iy) = -1.$$

Find the of images in $\mathbb{D}$ of the set of horozontal lines in S and the set of vertical line segments in S. Hint: In each case the set of images does not depend on the choice of f.

10. Let $I = [0,1]$ be the closed unit interval in $\mathbb{R}$ and let $U = \mathbb{C} \setminus I$.

(a) Prove there exists a non-constant bounded analytic function on U.

(b) If $f(z)$ is bounded and analytic on U and if f has a continuous extension to $\mathbb{C}$, prove f is constant.

11. Let $u(z)$ be an harmonic function on the complex plane $\mathbb{C}$ such that

$$\int\int_{\mathbb{C}} |u(z)|^2 dxdy < \infty.$$

Prove $u(z) = 0$ for all $z \in \mathbb{C}$.

12. Evaluate

$$\int_0^\infty \frac{x^2 dx}{x^4 + 6x^2 + 13}.$$

13. Let U be a domain in the complex plane such that $0 \in U$ and let $f : U \to U$ be an analytic function from U into U such that $f(0) = 0$ and $|f'(0)| < 1$. Let

$$f^{(n)}(z) = f \circ f \circ \cdots \circ f(z)$$

be the function obtained by composing f n-times. Prove that $f^{(n)}(z) \to 0 (n \to \infty)$ uniforml on compact subsets of U. Hint: First find a disc $B = \{|z| < a\} \subset U$ such that $f(B) \subset B$.

14. Let $S = [0,1] \times [0,1]$ be the unit square in $\mathbb{C}$ and let $f : S \to \mathbb{C}$ be a *continuous* map such that $f(z) \neq 0$ for all $z \in S$. Prove there exists continuous $g : S \to \mathbb{C}$ such that

$$f(z) = e^{g(z)}$$

for all $z \in S$.