Analysis Qualifying Examination - January 2002

Instructions:

Work any 12 problems. All problems are worth ten points.

1. Suppose f_n is a sequence of continuous functions on $[0,1]$ which converges to a continuous function f on $[0,1]$. Does it follow that f_n converges uniformly? Give a proof or provide a counterexample.

2. Let $\{a_n\}_n$ be a sequence of positive numbers converging to 0. Show that given any $x > 0$ there exist non-negative integers $k_1, k_2, \ldots$ such that $\Sigma_n k_n a_n = x$.

3. Let f and g be Lebesgue integrable functions on the interval $[0,1]$. Set

$$F(x) = \int_0^x f(y)dy, \; G(y) = \int_0^y g(x)dx$$

where dy and dx denote Lebeqgue measure. Assume $h(x,y) = f(y)g(x)$ is Lebesgue measureable on $[0,1] \times [0,1]$. Prove

$$\int_0^1 F(x)g(x)dx = F(1)G(1) - \int_0^1 f(y)G(y)dy.$$

4. Let $F(x)$ be a bounded real valued function on $\mathbb{R}$ such that

$$\lim_{x \to -\infty} F(x) = 0$$

and such that for all $\epsilon > 0$ there is $\delta > 0$ such that whenever $(a_j, b_j), 1 \leq j \leq n$ is a finite family of pairwise disjoint open intervals,

$$\sum (b_j - a_j) < \delta \Rightarrow \sum |F(b_j) - F(a_j)| < \epsilon.$$

Prove there is $f \in L^1(\mathbf{R})$ such that for all $x \in \mathbb{R}$,

$$F(x) = \int_{-\infty}^x f(t)dt.$$

5. Let $f : [0,1] \to \mathbb{C}$ be Lebesgue measurable. Assume $fg \in L^2([0,1], \mu)$ for every $g \in L^2([0,1], \mu)$, where μ is the Lebesgue measure on $[0,1]$. Prove that $f \in L^\infty([0,1], \mu)$.

6. Let $x \in \mathbb{R}$ have decimal expansion

$$x = a_0.a_1a_2a_3\ldots$$

and assume this expansion is unique (it is unique for almost all x). Define

$$A_n(x) = \begin{cases} 1, & \text{if } a_n \text{ is even} \\ -1, & \text{if } a_n \text{ is odd.} \end{cases}$$

Let $f \in L^1(\mathbb{R})$. Prove

$$\lim_{n\to\infty} \int_{\mathbf{R}} f(x)A_n(x)dx = 0.$$

7. Let $0 < p^* < \infty$. Let $n \geq 2$ and let μ be Lebesgue measure on $\mathbb{R}^n$. Prove there exists a Lebesgue measurable f on $\mathbb{R}^n$ such that for $0 < p < \infty$,

$$f \in L^p(\mathbb{R}^n, \mu) \Leftrightarrow p = p^*.$$

8. Let $\mathcal{H}$ be a Hilbert space.

a). Show that if $T : \mathcal{H} \to \mathcal{H}$ is a linear transformation such that $id_{\mathcal{H}} - T$ is a bounded operator on $\mathcal{H}$ with $\|id_{\mathcal{H}} - T\| < 1$, then T is a bounded, invertible operator on $\mathcal{H}$.

b). Assume $\{e_n\}_n \subset \mathcal{H}$ is an orthonormal basis for $\mathcal{H}$; that is $\{e_n\}$ is an orthonormal system, i.e. $< e_n, e_m >= \delta_{n,m}$ for all n, m, and $\overline{\text{span}}\{e_n\}_n = \mathcal{H}$. Show that if $\{f_n\}_n \subset \mathcal{H}$ is an orthonormal system such that $\Sigma_n \|e_n - f_n\|^2 < 1$ then $\{f_n\}_n$ is a basis for $\mathcal{H}$.

9. Find the Laurent series expansion for

$$\frac{1}{z(z+1)}$$

valid in $\{1 < |z-1| < 2\}$.

10. Prove, by using the residue calculus, that

$$\int_0^\infty \frac{\sqrt{x}}{1+x^2}dx = \pi\sqrt{2}$$

11. Let f and g be continuous functions on the real line related by Fourier transform, i.e.,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(k)e^{-ikx}dk, \qquad g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{ikx}dx$$

Prove that *both* f and g cannot be compactly supported (i.e., vanish outside some finite interval of the real line).

12. Let $f(z)$ be an entire function which is not a constant. Prove that $e^{f(z)}$ has an essential singularity at $z = \infty$.

13. Explain why there is a unique analytic function g in the domain D roughly sketched below (shaded region) with the property that $e^{g(z)} = z$ for all $z \in D$ and $g(1) = 2\pi i$. Compute $g(1 + i)$.