ANALYSIS QUALIFYING EXAM
FALL 2003

Directions. Each problem is worth 10 points. A complete solution to one problem is worth more than two half-solutions to two problems. You are to solve any 10 problems.

Problem 1. Let μ be a finite Borel measure on $[0,1]$. Suppose that

$$f : [0,1] \times [0,1] \to \mathbb{R}$$

is a Borel function, such that for each x, the map $t \mapsto f(x,t)$ is differentiable, and that $\left|\frac{\partial f}{\partial t}(x,t)\right| \leq g(x)$ for some Borel function $g(x)$ satisfying $\int_0^1 g(x)d\mu(x) < \infty$. Carefully prove that $F(t) = \int_0^1 f(x,t)d\mu(x)$ satisfies

$$F'(t) = \int_0^1 \frac{\partial f}{\partial t}(x,t)d\mu(x).$$

Problem 2. Let μ be a positive Borel measure on the unit interval $I = [0,1]$, such that $\mu(I) = 1$. Let $\xi_n(x) = x^n$, $n = 0, 1, 2, \ldots$.

(a) Let H be the Hilbert space $L^2([0,1], \mu)$. Show that $\xi_n \to 0$ in norm if and only if $\mu(\{1\}) = 0$.

(b) Let H be as in part (a). Show that if $f \perp \xi_n$ for all n, then f must be a.e. zero.

(c) Let V be the Banach space $L^\infty([0,1], \mu)$. Show that $\xi_n \to 0$ in norm if and only if for some $\varepsilon > 0$, $\mu([1-\varepsilon, 1]) = 0$.

Problem 3. Let $f \in L^1(\mathbb{R})$. Define its Fourier transform by the formula

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x) e^{2\pi i(xt)} dx.$$

Assume that both $f \in L^1(\mathbb{R})$ and $\hat{f} \in L^1(\mathbb{R})$. Show that

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(t) e^{-2\pi i(xt)} dt \qquad x \text{ a.e.}$$

Problem 4. Let H be a Hilbert space. Show that the unit sphere $S = \{\xi \in H : ||\xi|| = 1\}$ is compact (in the norm topology) if and only if H is finite-dimensional.

Problem 5. Let H be a Hilbert space. A sequence $\xi_n \in H$ is said to converge weakly to ξ if for all $\zeta \in H$, $\langle \xi_n, \zeta \rangle \to \langle \xi, \zeta \rangle$.

(a) Show that $||\xi|| \leq \limsup_n ||\xi_n||$;

(b) Give an example in which the inequality in (a) is strict;

(c) Show that if $||\xi|| = \lim_n ||\xi_n||$, then $||\xi - \xi_n|| \to 0$.

Problem 6. We let $\ell^p = L^p(\mathbb{N})$, where $\mathbb{N}$ has the usual counting measure, and we let $L^p = L^p([0,1])$ where $[0,1]$ has the usual Lebesgue measure. Show that $\ell^1 \subsetneqq \ell^2 . \subsetneqq \ell^\infty$, and that $L^1 \supsetneqq L^2 \supsetneqq L^\infty$.

Problem 7. Define the Haar functions on $[0,1]$ by

$$\begin{aligned} e_0(x) &= 1, \\ e_{n,k}(x) &= \begin{cases} 2^{\frac{n}{2}}, & \text{if } \frac{k-1}{2^n} \le x < \frac{k-\frac{1}{2}}{2^n} \\ -2^{\frac{n}{2}}, & \text{if } \frac{k-\frac{1}{2}}{2^n} \le x < \frac{k}{2^n} \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Show that these form an orthonormal basis for $L^2[0,1]$.

Problem 8. Let $I = [-1,1]$ be the closed unit interval in $\mathbb{R}$. Let $U = \mathbb{C} \setminus I$.

(a) Show that there exists a non-constant bounded analytic function on U.

(b) Prove that if $f(z)$ is bounded and analytic on U and if f has a continuous extension to $\mathbb{C}$, then f must be constant.

Problem 9. Let $A = \{z : \mathrm{Im} z > 0\} \setminus \{z : \mathrm{Re} z = 0 \text{ and } 0 \le \mathrm{Im} z \le 1\}$. Find a conformal map that maps A one-to-one onto the upper half place $\{z : \mathrm{Im} z > 0\}$, or show that no such map exists.

Problem 10. Use a contour integral to evaluate

$$\int_0^\infty \frac{1}{1+x^{2n}} dx, \qquad n \ge 1.$$

Problem 11. Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a complex polynomial. Show that there must be at least one point with $|z| = 1$ and $|p(z)| \ge 1$. (Hint: count zeros of $a_{n-1}z^{n-1} + \cdots + a_0$).

Problem 12. Let D be the open unit disk in the complex plane. Endow D with the Lebesgue measure λ. Let $A \subset L^2(D,\lambda)$ be the subspace consisting of those L^2 functions, which are analytic on the disk.

(a) Show that A is infinite-dimensional.

(b) Show that A is a closed subspace of $L^2(D,\lambda)$.