Analysis Qualifying Examination
January 11, 2003

Work any 10 problems, but include at least 3 problems from Part II. All problems s are worth 10 points, and a complete solution to one problem will be valued more highly than two half solutions to two problems.

Part I

1. Let μ be a finite, positive, regular Borel measure on $\mathbf{R}^2$, and let $\mathcal{G}$ be the family of finite unions of squares of the form

$$S = \{j2^n \leq x \leq (j+1)2^n; k2^n \leq y \leq (k+1)2^n\},$$

where j, k, and n are integers. Prove that the set of linear combinations of characteristic functions of elements of $\mathcal{G}$ is dense in $L^1(\mu)$.

2. Prove there is a constant C such that for every closed bounded interval $I = [a, b] \subset \mathbf{R}$ there is a constant α_I such that

$$\int_I \Big|\log|x| - \alpha_I\Big| dx \leq C(b-a).$$

3. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces and let $K(x, y)$ be measureable with respect to the product σ-algebra $\mathcal{M} \times \mathcal{N}$. Assume there is a constant $A > 0$ such that for all $x \in X$

$$\int_Y |K(x,y)| d\nu(y) \leq A,$$

and for all $y \in Y$,

$$\int_X |K(x,y)| d\mu(x) \leq A.$$

Let $1 \leq p \leq \infty$, and for $f \in L^p(X, \mathcal{M}, \mu)$ define

$$Tf(y) = \int_X f(x)K(x,y)d\mu(x).$$

Prove

$$||TF||_{L^p(\nu)} \leq A||f||_{L^p(\mu)}.$$

4. Prove or disprove: If F is a strictly increasing continuous map from the real line $\mathbf{R}$ onto itself and if $A \subset \mathbf{R}$ is Lebesgue measureable, then $f^{-1}(A)$ is Lebesgue measurable.

5. Is the Banach space ℓ^∞ of bounded complex sequences $a = \{a_n\}_{n=1}^\infty$ with the supremum norm $||a||_\infty = sup_n|a_n|$ separable? Prove your answer is correct.

6. Let X be a finite-dimensional real normed linear space with norm $|| \ ||$, and let

$$a_1, a_2, \ldots, a_n$$

be a vector space basis over $\mathbf{R}$ for X. For $x = \sum_{j=1}^{n} x_j a_j \in X$, write $||x||^* = \sum_{j=1}^{n} |x_j|$. Prove there is a constant $C > 0$ such that for all $x \in X$,

$$C^{-1}||x||^* \leq ||x|| \leq C||x||^*.$$

Hint. One inequality is easy; for the other use the Hahn-Banach theorem and induction.

7. Let X be an infinite-dimensional complete normed linear space over $\mathbf{R}$. Prove that every vector space basis for X is uncountable. *Hint.* Use Problem 7 to show finite-dimensional subspaces of X are closed.

8. Let $n \geq 2$, let H be the Hilbert space $L^2(\mathbf{R}^n)$ of square (Lebesgue) integrable function on $\mathbf{R}^n$ and let e be a fixed vector in $\mathbf{R}^n$, $e \neq 0$. Prove that the linear transformation $T : H \to H$ defined by

$$Tf(x) = f(x + e)$$

has no nonzero eigenvector.

Part II

9. Let D be the domain in the complex plane $\mathbf{C}$ that is the intersection of the two open disks centered at ± 1 whose boundary circles pass through $\pm i$. Find a conformal map f of D onto the open unit disk $\Delta = \{|w| < 1\}$ such that $f(i) = 1$ and $f(-i) = -1$. (You may express f as a composition of other specific maps.) What are the images of arcs of circles passing through $\pm i$ under your map f? (Justify your answer.)

10. Let

$$f_m(z) = \sum_{k=-m}^{m} \frac{1}{(z - m - ik)^2}, \qquad g_n(z) = \sum_{m=1}^{n} f_m(z).$$

Show that the sequence $\{g_n(z)\}_{n=1}^{\infty}$ converges normally to ∞ as $n \to \infty$. *Hint.* Look first at $g_n(0)$.

11. Show by contour integration that

$$\int_0^{2\pi} \frac{d\theta}{x + \cos\theta}\, d\theta = \frac{2\pi}{\sqrt{x^2 - 1}}, \qquad x > 1.$$

Determine for which complex values of z the integral

$$\int_0^{2\pi} \frac{d\theta}{z + \cos\theta}\, d\theta$$

exists and evaluate the integral. Justify your reasoning.

12. Let S be a sequence of points in the complex plane that converges to 0. Let $f(z)$ be defined and analytic on some disk centered at 0 except possibly at the points of S and at 0. Show that either $f(z)$ extends to be meromorphic in some disk containing 0, or else for any complex number w there is a sequence $\{\zeta_j\}$ such that $\zeta_j \to 0$ and $f(\zeta_j) \to w$.