Analysis Qualifying Exam, September 2004.

Attempt any ten of the twelve questions.
Each question is worth 10 points.

Q1. For each natural number n, let $f_n : [0, 1] \to \mathbf{R}$ be a sequence of absolutely integrable functions, and let $f : [0, 1] \to \mathbf{R}$ be another absolutely integrable function such that

$$\int_0^1 |f_n(x) - f(x)| \, dx \to 0$$

as $n \to 0$.

(a) Show that there exists a subsequence f_{n_j} of f_n which converges to f pointwise almost everwhere.

(b) Give a counterexample to show that the assertion fails if "pointwise almost everywhere" is replaced by "uniformly".

Q2. Let $f : [0, 1] \to \mathbf{R}^+$ be a non-negative, absolutely integrable function. Prove that the following two statements are equivalent.

(i) There exists a constant $0 < C < \infty$ such that

$$\left(\int_{[0,1]} f(x)^p \, dx \right)^{1/p} \leq Cp \quad \text{for all } 1 \leq p < \infty.$$

(ii) There exists a constant $0 < c < \infty$ such that

$$\int_{[0,1]} e^{cf(x)} \, dx < \infty.$$

Q3. Let E be a measurable subset of the real line.

(a) Let $\chi_E : \mathbf{R} \to \mathbf{R}$ be the characteristic function of E (i.e. $\chi_E(x) = 1$ when $x \in E$ and $\chi_E(x) = 0$ when $x \notin E$. If E has finite Lebesgue measure, show that the function $f : \mathbf{R} \to \mathbf{R}$ defined by

$$f(x) := \int_{\mathbf{R}} \chi_E(y)\chi_E(y-x)\, dy$$

is continuous.

(b) Suppose instead that E has positive Lebesgue measure $0 < m(E) \leq \infty$. Using (a), show that the set $E - E := \{x - y : x, y \in E\}$ contains an open interval $(-\varepsilon, \varepsilon) := \{x \in \mathbf{R} : -\varepsilon < x < \varepsilon\}$ for some $\varepsilon > 0$.

Q4. Let $f : \mathbf{R} \to \mathbf{R}$ be a measurable bijective function such that

$$f(x) + f(y) = f(x+y)$$

for all $x, y \in \mathbf{R}$.

(a) Show that f is continuous.
Hint: Show that the set $E := f^{-1}((-\varepsilon, \varepsilon)) := \{x \in \mathbf{R} : -\varepsilon < f(x) < \varepsilon\}$ obeys the hypotheses of Q3(b) for each $\varepsilon > 0$.

(b) Show that there exists a non-zero real number c such that $f(x) = cx$ for all real numbers $x \in \mathbf{R}$.
Hint: work on the rational numbers $\mathbf{Q}$ first.

Q5. Let X be a Banach space endowed with the norm $\| \|_X$, let V be a closed subspace of X, and let $T : X \to X$ be a bounded linear transformation on X which has V as an invariant subspace (i.e. $Tx \in V$ for all $x \in V$). Let X/V be the quotient space $X/V := \{x + V : x \in X\}$ of translates of V, endowed with the norm

$$\|x + V\|_{X/V} := \inf\{\|y\|_X : y \in x + V\}.$$

You may assume without proof that X/V is a Banach space with this norm.

Let $T_1 : V \to V$ be the restriction of T to V, i.e. $T_1 x := Tx$ for all $x \in V$, and let $T_2 : X/V \to X/V$ be the map $T_2(x + V) := (Tx) + V$. You may assume without proof that T_1 and T_2 are well-defined bounded linear transformations.

Suppose that T_1 and T_2 are boundedly invertible on V and X/V respectively, i.e. there exists bounded linear operators $S_1 : V \to V$ and $S_2 : X/V \to X/V$ such that $S_1T_1 = T_1S_1$ is the identity on V, and $S_2T_2 = T_2S_2$ is the identity on X/V. Prove that T is also boundedly invertible.

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Q6. Let (X, d) be a compact metric space, and let $\mathcal{F}$ be the set of non-empty compact subsets of X. If K_1, K_2 are two elements of $\mathcal{F}$ (i.e. two non-empty compact subsets of X), define the *Hausdorff distance* $d_H(K_1, K_2)$ to be the quantity

$$d_H(K_1, K_2) := \max\Big(\sup_{x\in K_1} \inf_{y\in K_2} d(x, y),\ \sup_{y\in K_2} \inf_{x\in K_1} d(x, y) \Big).$$

You may assume without proof that $(\mathcal{F}, d_H)$ is a metric space. Show that $(\mathcal{F}, d_H)$ is complete.
Hint: It may help to keep the following intuition in mind: if $d_H(K_1, K_2) \le \varepsilon$, this means that K_1 lies in the ε-neighborhood of K_2, and K_2 lies in the ε-neighborhood of K_1.

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Q7. For each integer n, let $f_n : [0, 1] \to \mathbf{R}$ be an everywhere differentiable (and hence continuous) function. Suppose that the sequence of functions f_n converges pointwise, and that the sequence of functions f'_n converges uniformly. Prove that the sequence of functions f_n converges uniformly, and that the limit is everywhere differentiable.
Hints: the fundamental theorem of calculus requires that f_n be *continuously* differentiable, and so a proof using that theorem may only earn partial credit. However, the mean value theorem is valid for functions which are merely differentiable, and may be used without proof.

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Q8. Evaluate the integral

$$I := \int_{-\infty}^{+\infty} \frac{\cos x}{(1 + x^2)^2}\, dx.$$

You will need to briefly justify any limits you take.

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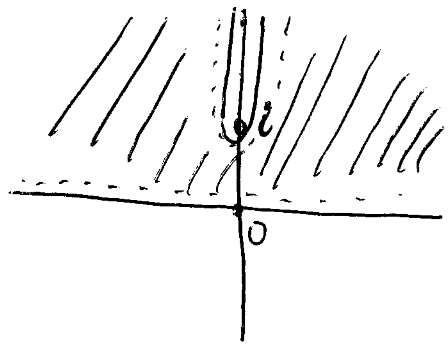
Q9. Find the number of zeros of the polynomial $z^6 + 12z^4 + z^3 + 2z + 6$ in the first quadrant inside the unit circle (the domain $\{z = x + iy : x > 0, y > 0, |z| < 1\}$), and also in the first quadrant outside the unit circle (the domain $\{z = x + iy : x > 0, y > 0, |z| > 1\}$).

Q10. Let $D := \{z = x + iy : y > 0\} \backslash [i, i\infty]$ be the domain obtained from the open upper half-plane by excising the vertical ray from i to $i\infty$, that is,



Find a conformal map $w = f(z)$ of D onto the open unit disk $\{|w| < 1\}$. You may represent the map $f(z)$ as a composition of other maps. Include a sketch or diagram with your solution.

Q11. Let $f(z)$ be a bounded analytic function on the open right half-plane $\{z = x + iy : x > 0\}$ such that $f(x) \to 0$ as x tends to 0 along the positive real axis. Suppose $0 < \theta_0 < \pi/2$. Prove that $f(z) \to 0$ as $z \to 0$, uniformly in the sector $|\arg z| \leq |\theta_0|$.

Q12. Let D be a domain, i.e. a connected open set. Let $f : D \to \mathbf{R}$ be a continuous function with the property that whenever $\overline{B(z_0, r)} := \{z : |z - z_0| \leq r\}$ is a closed disk contained in D, and $h : \overline{B(z_0, r)} \to \mathbf{R}$ is a harmonic function such that $f(z) \leq h(z)$ for all $z \in \partial B(z_0, r) := \{z : |z - z_0| = r\}$, we have $f(z_0) \leq h(z_0)$. (Such functions are called *subharmonic*).

Show that f cannot attain its maximum at any point in D unless it is a constant function.
