• UCLA Department of Mathematics •
Graduate Analysis Exam Winter 2004

No Smoking
No Drinking

There are altogether twelve porblems – six from "real analysis" and six from "complex analysis". Attempt all problems. Start each problem on a new page. Be sure to label your problems.

[A1] ———

Let $f: \mathbb{R}^n \to [0,B]$ denote a bounded function about which you may make no additional assumptions. For any $\varepsilon > 0$, let us define $\phi_\varepsilon(x)$ by

$$\phi_\varepsilon(x) = \sup_{y:|x-y|<\varepsilon} f(y).$$

(a) Show that the $\varepsilon \to 0$ limit of $\phi_\varepsilon(x)$ exists.

(b) Denoting this limit by $\phi(x)$, show that ϕ is upper–semicontinuous.

[A2] ———

Let $f: [0,1] \to (0,1)$ be a function of class $\mathscr{C}^1$ – that is to say both f and f′ are continuous functions – with

$$\max_{0 \le x \le 1} |f'(x)| \le 1-\varepsilon$$

for some $\varepsilon > 0$.

(a) Show that there is exactly one solution to the equation

$$f(x^*) = x^*.$$

(b) Let x_o denote any point in [0,1]. Define $x_1 = f(x_o)$ and, in general,

$$x_{n+1} = f(x_n).$$

Show that regardless of the choice of x_o, we have

$$\lim_{n\to\infty} x_n = x^*.$$

[A3] ———

Let $f: [0,1] \to [0,1]$ denote a Borel function and consider the subset of the square, A_f, defined by

$$A_f = \{(x,y) \in [0,1]\times[0,1] \mid y \le f(x)\}.$$

Show that A_f is a Borel set and that in fact,

$$m_2(A_f) = \int_{[0,1]} f(x)\,dm_1$$

where m_1 and m_2 denote one and two dimensional Lebesgue measure respectively.

[A4]

Let $\langle \mathbf{X}, \mu, \mathcal{B} \rangle$ denote a finite measure space and let $f \in L^\infty(d\mu)$. Define

$$\alpha_n = \int_{\mathbf{X}} |f|^n d\mu .$$

Show that

$$\lim_{n \to \infty} \frac{\alpha_{n+1}}{\alpha_n} = \|f\|_\infty .$$

[A5]

Let μ denote a finite Borel measure supported on a countable set $\mathbf{Q} \subset \mathbb{R}$ and let

$$F(t) = \int_{-\infty}^{+\infty} e^{ixt} d\mu(x)$$

denote its Fourier transform. Show that

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} |F(t)|^2 dt = \sum_{q \in \mathbf{Q}} |\mu(q)|^2 .$$

[A6]

Consider the Hilbert space ℓ^2 of all complex valued square–summable sequences. Let $\mathbf{T}$ be the operator that shifts the sequence to the right and places a zero in the first slot:

$$\mathbf{T}(\zeta_1, \zeta_2, \zeta_3, \dots) = (0, \zeta_1, \zeta_2, \zeta_3, \dots).$$

(a) For any $\underline{a} \in \ell^2$, let $\underline{a}^{[n]} = \mathbf{T}^n \underline{a}$. Show that the $\underline{a}^{[n]}$ converge weakly to zero, i.e. that for any $\underline{b} \in \ell^2$, the numbers $\langle \underline{a}^{[n]} | \underline{b} \rangle$ converge to zero.

(b) Compute the adjoint, $\mathbf{T}^*$, of $\mathbf{T}$ and show that for any $\underline{a} \in \ell^2$, the vectors $[\mathbf{T}^*]^n \underline{a}$ converge *strongly* (i.e. in norm) to zero.

[CA 1]

Let $f(z) = u(x,y) + iv(x,y)$ denote a non–constant analytic function on some open domain $D \subset \mathbb{C}$. Show that at each point of D, the "level curves" $u(x,y) =$ constant and $v(x,y) =$ constant intersect at right angles.

[CA 2]

Let $K(z)$ denote a real–valued function of the complex variable z defined in some open domain $D \subset \mathbb{C}$. Then $K(z)$ is said to be *strictly* subharmonic if the inequality

$$K(z_o) < \frac{1}{2\pi}\int_0^{2\pi} K(z_o + \rho e^{i\phi})d\phi$$

holds for all ρ which are smaller than the distance from z_o to the boundary of D. Let $f(z)$ denote a *non–constant* analytic function on D. Show that $|f(z)|$ is strictly subharmonic.

[CA 3]

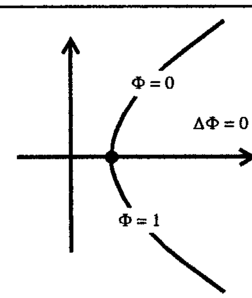
A function $f(z)$ is entire and has the property that for some fixed $\lambda > 0$, the inequality

$$|\text{Re}[f(z)]| \geq \lambda|\text{Im}[f(z)]|$$

holds for all $z \in \mathbb{C}$. Show that this function must be a constant.

[CA 4]

A bounded harmonic function $\Phi(x,y)$ is defined in the region of the plane $x^2 \geq y^2 + 1$. The function Φ satisfies the boundary condition that for $x^2 - y^2 = 1$ with $y > 0$, $\Phi = 0$ while for $x^2 - y^2 = 1$ with $y < 0$, $\Phi = 1$. Find this function. You may express your answer in terms real and/or imaginary parts of well known analytic functions.



[CA 5]

Using contour (or any other) methods, compute the integral

$$\int_0^\infty \frac{x dx}{1 + x^3}.$$

Hint: Find a "path of return" where you compute an integral related to the one you want.

[CA 6]

Let $f(z)$ denote a function which is analytic at all points on a simple closed curve, γ, and everywhere inside γ except for the (isolated) points $b_1, b_2, \ldots b_r$ where it has poles of order $\beta_1, \beta_2, \ldots \beta_r$. Furthermore, f does not vanish on γ but does have zeros at the points $a_1, a_2, \ldots a_s$ which are inside γ and these zeros are of multiplicity $\alpha_1, \alpha_2, \ldots \alpha_s$. Then, according to a well known formula,

$$\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}dz = \sum_{k=1}^{s}\alpha_k - \sum_{k=1}^{r}\beta_k .$$

Now suppose that $g(z)$ is analytic inside and on γ. What is the generalization of the above formula when the integrand on the left hand side is replaced by $g(z)\frac{f'(z)}{f(z)}$? Provide justification for your answer.